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Lie–Rinehart Algebras

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The present Master thesis “Lie–Rinehart Algebras” has been realized under my supervision by the graduate student Xabier García Martínez as his Master’s thesis inside the “Máster Universitario en Matemáticas” of the University of Santiago de Compostela, and I authorize its presentation.

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Manuel Ladra González

Contents

Abstract

In this thesis we collect the main definitions and results of Lie–Rinehart algebras and then we present our recent work on universal central extensions and a non-abelian tensor product of Lie–Rinehart algebras. We start with definitions, examples and some constructions. Then we see the relations between Lie–Rinehart algebras and Poisson algebras. We define the universal enveloping algebra and we explore some of its properties, giving a proof of the version of the PBW theorem in Lie–Rinehart algebras and we see that it has a left Hopf algebroid structure. We also define the Lie–Rinehart superalgebras and the Restricted Lie–Rinehart algebras structures. Then we introduce the Lie–Rinehart (co)homology. To end the thesis, we present our main results of universal central extensions in Lie–Rinehart algebras and the definition of the non-abelian tensor product.

Resumen

En este trabajo recopilamos las principales definiciones y resultados sobre álgebras de Lie–Rinehart y luego presentamos nuestro reciente trabajo sobre extensiones centrales universales y un producto tensorial no abeliano. Empezamos con definiciones, ejemplos y algunas construcciones. Luego vemos la relación entre álgebras de Lie–Rinehart y álgebras de Poisson. Definimos el álgebra envolvente universal y exploramos algunas de sus propiedades, dando una prueba de la versión del teorema PBW en álgebras de Lie–Rinehart y vemos que tiene una estructura de algebroide de Hopf por la izquierda. También definimos las estructuras de superálgebras de Lie–Rinehart y álgebras de Lie–Rinehart restringidas. Luego introducimos la (co)homología de Lie-Rinehart. Por último, presentamos nuestros principales resultados en extensiones centrales universales en álgebras de Lie–Rinehart y la definición del producto tensorial no abeliano.

Introduction

The concept of a Poisson manifold is currently of much interest in mathematics and physics. A key idea is that a Poisson structure $[\cdot,\cdot]$ on an arbitrary algebra A over a commutative ring K gives rise to a structure of a Lie–Rinehart algebra over A in the sense of Rinehart [28] on the A-module Ω^1_{A} of Kähler differentials for A in a natural fashion.

A Lie–Rinehart algebra over A is a Lie algebra over R which acts on A by derivations and it is also an A-module satisfying suitable compatibility conditions which generalize the usual properties of the Lie algebra of smooth vector fields on a smooth manifold viewed as a module over its ring of smooth functions; these objects have been introduced by Herz [12] under the name "pseudo-algèbre de Lie" and were examined by Palais [26] under the name "d-Lie ring".

Any Lie–Rinehart algebra L over A gives rise to a complex $\mathrm{Alt}_{\mathrm{A}}(\mathrm{L},\mathrm{A})$ of alternating forms which generalizes the usual de Rham complex of a manifold and the usual complex computing Chevalley-Eilenberg [7] Lie algebra cohomology. This observation is again due to Palais [26]. Moreover, extending earlier work of Hochschild, Kostant and Rosenberg [13], Rinehart [28] has shown that, when L is projective as an A-module, the homology of the complex $\mathrm{Alt}_{\mathrm{A}}(\mathrm{L},\mathrm{A})$ may be identified with $\mathrm{Ext}^*_{\mathsf{U}_{\mathrm{A}}\mathrm{L}}(\mathrm{A},\mathrm{A})$ over a suitably defined universal algebra $\mathsf{U}_{\mathrm{A}} L$ of differential operators. In particular, when A is the algebra of smooth functions on a smooth manifold M and L the Lie algebra of smooth vector fields on M, then $\mathsf{U}_{\mathrm{A}} L$ is the algebra of (globally defined) differential operators on M.

The concept of an Lie–Rinehart algebra over A has a geometric analogue which is nowadays called a *Lie algebroid*, see Cannas-Weinstein [2], Coste-Dazord-Weinstein [8], Mackenzie [23], Pradines [27] or Weinstein [30].

Moreover, there are some structures generalizing Lie–Rinehart algebras as Lie–Rinehart superalgebras studied by Chemla [6], Leibniz–Rinehart algebras studied by Ibañez-León-Marrero [16] or restricted Lie–Rinehart algebras studied by Dokas [9].

This thesis is divided in 4 chapters. On the first one, we will introduce Lie–Rinehart algebras, the category they form and several examples to have an idea about they behaviour.

Then we will define some structures as the semidirect product, introduced by Rinehart [28], the left Lie–Rinehart (A, L)-modules introduced by Palais [26], right Lie–Rinehart (A, L)-modules introduced by Huebschmann [14] and crossed modules introduced by Casas-Ladra-Pirashvili [4]. Then we will show the connections with Poisson algebras, as we can see in Loday-Vallette [22], and in particular with Kähler differentials studied García-Beltrán, Vallejo and Vorobjev [11].

In the second chapter, we will give the definition of the universal enveloping algebra of a Lie–Rinehart algebra first given by Rinehart [28] and we will explore some properties given by Moerdijk-Mrčun [24]. Then we will prove a version of the Poincaré-Birkhoff-Witt theorem for Lie–Rinehart algebras first proved by Rinehart [28] and we will see that the universal enveloping algebra has a canonical left Hopf algebroid structure studied by Kowalzig [19]. To end the chapter we will see two generalizations of Lie–Rinehart algebras and their own universal enveloping algebra. These structures are Lie–Rinehart superalgebras introduced by Chemla [6] and Restricted Lie–Rinehart algebras studied by Dokas [9]. All the sections in this chapter will begin with some results in Lie algebras in order to generalize them to Lie–Rinehart algebras.

In chapter three, we will see some classical results on (co)homology theory in Lie algebras that can be found in [21] and [29] in order to generalize them to the (co)homology theory of Lie–Rinehart algebras, studied by Huebschmann [14] and Rinehart [28].

The final chapter will be the main part of this thesis. We will expose our recent work in relation with universal central extensions. We will prove that the existence of a universal central extension of a Lie–Rinehart algebra L is equivalent that L being perfect. Then we will give a explicit construction of a functor $\mathfrak{uce}_{\mathrm{A}}$ from the category of Lie–Rinehart algebras to itself which in the case of L being perfect, $\mathfrak{uce}_{\mathrm{A}} L$ will be the universal central extension of L. To end the chapter, we will introduce a generalization on Lie–Rinehart algebras of the non-abelian tensor product of Lie algebras introduced by Ellis [10], we will find some properties and we will relate it to the universal central extension.

Chapter 1

Lie–Rinehart Algebras

1.1 Preliminaries

In this section we will give some basic definitions of the topic and some examples. We remark that if nothing else is said, all tensor product $\otimes$, will be tensor products over K, $\otimes_K$.

Definition 1.1. Let K be a commutative unital ring and A a commutative unital algebra over K. A *K-derivation*, is a K-linear map $D\colon \mathrm{A} \to \mathrm{A}$ which satisfies the Leibniz's law

$$D(ab) = (Da)b + a(Db),$$

for all $a, b \in \mathrm{A}$.

Then the set $\mathtt{Der}_K(\mathrm{A})$ of all K-derivations of A is a Lie K-algebra with Lie bracket $[D, D'] = DD' - D'D$, and an A-module simultaneously. These two structures are related by the following identity

$$[D, aD'] = a[D, D'] + D(a)D', \quad D, D' \in \mathtt{Der}_K(\mathrm{A}).$$

This leads to the notion below, which goes back to Herz under the name "pseudo–algèbre de Lie" in [12].

Definition 1.2. Let A be a commutative, unital algebra over a commutative unital ring K. A *Lie–Rinehart algebra over* A is a K-Lie algebra L with an A-module structure and a map (usually called anchor)

$$\alpha_L\colon L \to \mathtt{Der}_K(\mathrm{A}),$$

which is simultaneously a Lie algebra and A-module homomorphism and the K-Lie algebra structure and the A-module structure on L are related by the identity

$$[x, ay] = a[x, y] + x(a)y,$$

where $a \in A$, $x, y \in L$ and by $x(a)$ we mean $\alpha_L(x)(a)$. These objects are also known as (K, A)-Lie algebras [28] and d-Lie rings [26].

In this way, we see that $\mathtt{Der}_K(\mathrm{A})$ is a Lie–Rinehart algebra over A, where $\alpha_{\mathtt{Der}_K(\mathrm{A})} = \mathsf{Id}_{\mathtt{Der}_K(\mathrm{A})}$.

Example 1.3. Let us observe that Lie–Rinehart algebras over A with trivial anchor map $\alpha_L \colon L \to \mathtt{Der}_K(\mathrm{A})$ are exactly Lie A-algebras. If $\mathrm{A} = K$, then $\mathtt{Der}_K(\mathrm{A}) = 0$ and there is no difference between Lie and Lie–Rinehart algebras. Therefore the concept of Lie–Rinehart algebras generalizes the concept of Lie A-algebras.

Before having more examples of Lie–Rinehart algebras we will see how is the category that they form.

Definition 1.4. Let L and M Lie–Rinehart algebras over A. We say that $f \colon L \to M$ is a *Lie–Rinehart homomorphism over* A if it is simultaneously a K-Lie algebra and A-module homomorphism. Furthermore f must conserve the action in $\mathtt{Der}_K(\mathrm{A})$, in other words, the following diagram must be commutative.

$$\begin{tikzcd} L \arrow[rr, "f"] \arrow[dr, "\alpha_L"'] & & M \arrow[dl, "\alpha_M"] \\ & \mathtt{Der}_K(\mathrm{A}) & \end{tikzcd}$$

We denote by $\mathtt{LR}_{\mathrm{AK}}$ the category of Lie–Rinehart algebras over A. By Example 1.3 we have the full inclusion

$$\mathtt{Lie}_{\mathrm{A}} \subset \mathtt{LR}_{\mathrm{AK}},$$

where $\mathtt{Lie}_{\mathrm{A}}$ denotes the category of A-Lie algebras.

It is important to recall that the product in the category $\mathtt{LR}_{\mathrm{AK}}$ as a set, is not the product in $\mathtt{Set}$. Given two Lie–Rinehart algebras L and M, the product in $\mathtt{LR}_{\mathrm{AK}}$ is $L \times_{\mathtt{Der}_K(\mathrm{A})} M = \{(x, m) \in L \times M : x(a) = m(a) \text{ for all } a \in \mathrm{A}\}$, with the obvious action $(x, m)(a) = x(a) = m(a)$ for all $a \in \mathrm{A}$, $x \in L$ and $m \in M$.

Remark 1.5. Some authors consider a different category when they speak about Lie–Rinehart algebras. One can define the category changing the base K-algebra, so the objects will be pairs of the form (A, L) and the morphisms will be pairs (ϕ, f), where $\phi \colon \mathrm{A} \to \mathrm{A}'$ is a homomorphism of K-algebras, $f \colon L \to L'$ a Lie algebra homomorphism and they are related by $f(ax) = \phi(a)f(x)$ and $\phi\big(x(a)\big) = f(x)\big(\phi(a)\big)$. In the most part of topics that we are going to study here, it is irrelevant which category are we considering, but by default we are not going to change our base K-algebra, so we will be in the category defined above.

Definition 1.6. Let L be a Lie–Rinehart algebra over A. A *Lie–Rinehart subalgebra* M of L is a K-Lie subalgebra which is an A-module, with action induced by the inclusion in L. If M and N are two Lie–Rinehart subalgebras of L, we define the *commutator* of M and N, denoted by $\{M, N\}$ as the A-module spanned by the elements of the form $a[x, y]$ where $a \in \mathrm{A}$, $x \in M$ and $y \in N$.

Definition 1.7. Given a subalgebra M of L we say that it is a *Lie–Rinehart ideal* if M is a K-Lie ideal of L and the action induced by the inclusion is the trivial action, i.e. $\alpha(M) = 0$.

An example of an ideal is the kernel of a Lie–Rinehart homomorphism. Another example is the center of a Lie–Rinehart algebra, defined by

$$\mathsf{Z}_{\mathrm{A}}(L) = \{x \in L : [ax, z] = 0 \text{ and } x(a) = 0 \text{ for all } a \in \mathrm{A}, z \in L\}.$$

We denote by L^{ab} the A-module $L/\{L, L\}$. We recall that the subalgebra $\{L, L\}$ is not necessarily a Lie–Rinehart ideal, so L^{ab} may not be a Lie–Rinehart algebra.

We will see now more examples of Lie–Rinehart algebras.

Example 1.8. If $\mathfrak{g}$ is a K-Lie algebra acting on a commutative K-algebra A by derivations (that is, a homomorphism of Lie K-algebras $\gamma\colon \mathfrak{g} \to \mathtt{Der}_K(\mathrm{A})$ is given), then the *transformation Lie–Rinehart algebra of* $(\mathfrak{g}, \mathrm{A})$ is $L = \mathrm{A} \otimes \mathfrak{g}$ with the Lie bracket

$$[a \otimes g, a' \otimes g'] := aa' \otimes [g, g'] + a\gamma(g)(a') \otimes g' - a'\gamma(g')(a) \otimes g,$$

where $a, a' \in \mathrm{A}$, $g, g' \in \mathfrak{g}$ and the action $\alpha_L\colon L \to \mathtt{Der}_K(\mathrm{A})$ is given by $\alpha_L(a \otimes g)(a') = a\gamma(g)(a')$.

Example 1.9. Let $\mathcal{M}$ be an A-module. The *Atiyah algebra* $\mathcal{A}_{\mathcal{M}}$ of $\mathcal{M}$ is the Lie–Rinehart A-algebra whose elements are pairs (f, D) with $f \in \mathrm{End}_K(\mathcal{M})$ and $D \in \mathtt{Der}_K(\mathrm{A})$ satisfying the following property:

$$f(am) = af(m) + D(a)m, \quad a \in \mathrm{A}, m \in \mathcal{M}.$$

$\mathcal{A}_{\mathcal{M}}$ is a Lie–Rinehart A-algebra with the Lie bracket

$$[(f, D), (f', D')] = ([f, f'], [D, D'])$$

and anchor map $\alpha_{\mathcal{A}_{\mathcal{M}}}(f, D) = D$.

Example 1.10. Consider the K-algebra of dual numbers,

$$\mathrm{A} = K[\varepsilon] = K[X]/(X^2) = \big\{c_1 + c_2\varepsilon \mid c_1, c_2 \in K, \varepsilon^2 = 0\big\}.$$

We can endow to A with the Lie algebra structure given by the bracket:

$$[c_1 + c_2\varepsilon, c_1' + c_2'\varepsilon] = (c_1c_2' - c_2c_1')\varepsilon, \quad c_1 + c_2\varepsilon, c_1' + c_2'\varepsilon \in \mathrm{A}.$$

Thus A is a Lie–Rinehart algebra over A with anchor map $\alpha_{\mathrm{A}} \colon \mathrm{A} \to \mathtt{Der}_K(\mathrm{A})$, $c_1 + c_2\varepsilon \mapsto \mathtt{ad}_{c_1}$, where $\mathtt{ad}_{c_1}(c_1' + c_2'\varepsilon) = [c_1, c_1' + c_2'\varepsilon]$ is the adjoint map of c_1.

Example 1.11. The A-module $\mathtt{Der}_K(\mathrm{A}) \oplus \mathrm{A}$ is a Lie–Rinehart algebra over A with the bracket

$$[(D, a), (D', a')] = \big([D, D'], D(a') - D'(a)\big),$$

and anchor map $\pi_1 \colon \mathtt{Der}_K(\mathrm{A}) \oplus \mathrm{A} \to \mathtt{Der}_K(\mathrm{A})$, the projection onto the first factor.

Definition 1.12. Let M be a smooth manifold. Denote by $T(\mathrm{M})$ the tangent bundle of M and $\mathfrak{X}(\mathrm{M})$ the Lie algebra of smooth vector fields on M. A *Lie algebroid* over M is a real smooth vector bundle $\pi \colon \mathfrak{g} \to \mathrm{M}$ over M, together with a smooth map $\mathrm{an} \colon \mathfrak{g} \to T(\mathrm{M})$ of vector bundles over M and a Lie algebra structure on the vector space $\Gamma\mathfrak{g}$ of smooth sections of $\mathfrak{g}$, such that

1. the induced map $\Gamma(\mathrm{an}) \colon \Gamma\mathfrak{g} \to \mathfrak{X}(M)$ is a Lie algebra homomorphism,
2. the identity
$$[x, fy] = f[x, y] + \Gamma(\mathrm{an})(x)(f)y,$$
holds for any $f \in \mathcal{C}(\mathrm{M})$ and $x, y \in \Gamma\mathfrak{g}$.

Example 1.13. In particular, let $K = \mathbb{R}$ and $\mathrm{A} = \mathcal{C}^\infty(\mathrm{M})$ be the algebra of smooth functions on a compact manifold M and let L be a Lie–Rinehart algebra over A. Assume that L is finitely generated and projective as an A-module. Then it follows from Serre–Swan's theorem that $L = \mathcal{C}^\infty(E)$, is the space of smooth sections of a vector bundle over M. The bundle map $\alpha \colon E \to T(\mathrm{M})$ induces $\alpha \colon \mathcal{C}^\infty(E) \to \mathtt{Der}_{\mathbb{R}}\big(\mathcal{C}^\infty(\mathrm{M})\big) = \mathcal{C}^\infty\big(T(\mathrm{M})\big)$. In other words, Lie algebroids over M are precisely the Lie–Rinehart algebras over $\mathcal{C}^\infty(\mathrm{M})$ which are finitely generated and projective as $\mathcal{C}^\infty(\mathrm{M})$-modules. So one recovers Lie algebroids as a particular case of Lie–Rinehart algebras.

1.2 Constructions and Actions

Definition 1.14. Let $L \in \mathtt{LR}_{\mathrm{AK}}$ and let R be a Lie A-algebra. We will say that L *acts on* R if it is given a K-linear map

$$L \otimes R \to R,\ (x, r) \mapsto x \circ r,\ x \in L, r \in R$$

such that the following identities hold

1) $[x,y]\circ r = x\circ(y\circ r) - y\circ(x\circ r)$,

2) $x\circ[r,r'] = [x\circ r, r'] - [x\circ r', r]$,

3) $ax\circ r = a(x\circ r)$,

4) $x\circ(ar) = a(x\circ r) + x(a)r$,

where $a\in \mathrm{A}$, $x,y\in L$ and $r,r'\in R$.

Let us observe that 1) and 2) mean that L acts on R in the category of Lie K-algebras.

Definition 1.15. Let us consider a Lie–Rinehart algebra L and a Lie A-algebra R on which L acts. Since L acts on R in the category of Lie K-algebras as well, we can form the *semi–direct product* $L\rtimes R$ in the category of Lie K-algebras, which is $L\oplus R$ as a K-module, equipped with the following bracket

$$[(x,r),(y,r')] := ([x,y],[r,r'] + x\circ r' - y\circ r),$$

where $x,y\in L$ and $r,r'\in R$. We claim that $L\rtimes R$ has also a natural Lie–Rinehart algebra structure. Firstly, $L\rtimes R$ as an A-module is the direct sum of A-modules L and R. Hence $a(x,r) = (ax,ar)$. Secondly the map

$$\widetilde{\alpha}\colon L\rtimes R \to \mathtt{Der}_K(\mathrm{A})$$

is given by $\widetilde{\alpha}(x,r) := \alpha_L(x)$. In this way we really get a Lie–Rinehart algebra. Indeed, it is clear that $\widetilde{\alpha}$ is simultaneously an A-module and a Lie algebra homomorphism and we obtain

$$\begin{aligned}
[(x,r),a(y,r')] &= [(x,r),(ay,ar')] = ([x,ay],[r,ar'] + x\circ(ar') - ay\circ r)\\
&= \big(a[x,y] + x(a)y, a[r,r'] + a(x\circ r') + x(a)r' - a(y\circ r)\big)\\
&= a([x,y],[r,r'] + x\circ r' - y\circ r) + (x(a)y, x(a)r')\\
&= a[(x,r),(y,r')] + x(a)(y,r').
\end{aligned}$$

Thus $L\rtimes R$ is indeed a Lie–Rinehart algebra.

Definition 1.16. *A left Lie–Rinehart* (A,L)*-module* over a Lie–Rinehart A-algebra L is a K-module $\mathcal{M}$ together with two operations

$$L\otimes\mathcal{M}\to\mathcal{M},\qquad (x,m)\mapsto xm,$$

and

$$\mathrm{A}\otimes\mathcal{M}\to\mathcal{M},\qquad (a,m)\mapsto am,$$

such that the first one makes $\mathcal{M}$ into a module over the Lie K-algebra L in the sense of the Lie algebra theory, while the second map makes $\mathcal{M}$ into an A-module and additionally the following compatibility conditions hold

$$(ax)(m) = a(xm),$$
$$x(am) = a(xm) + x(a)m, \qquad a \in \mathrm{A}, m \in \mathcal{M} \text{ and } x \in L.$$

Notice that a left Lie–Rinehart (A, L)-module is equivalent to giving a morphism of Lie–Rinehart A-algebras $L \to \mathcal{A}_{\mathcal{M}}$ (see Example 1.9).

Definition 1.17. For a left Lie-Rinehart (A, L)-module $\mathcal{M}$ one can define *the semi-direct product* $L \rtimes \mathcal{M}$ to be $L \oplus \mathcal{M}$ as an A-module with the bracket $[(x, m), (y, n)] = \big([x, y], xn - ym\big)$, $x, y \in L, m, n \in \mathcal{M}$.

Definition 1.18. *A right Lie–Rinehart* (A, L)*-module* over a Lie–Rinehart A-algebra L is a K-module $\mathcal{M}$ together with two operations

$$\mathcal{M} \otimes L \to \mathcal{M}, \qquad (m, x) \mapsto mx,$$

and

$$\mathrm{A} \otimes \mathcal{M} \to \mathcal{M}, \qquad (a, m) \mapsto am,$$

such that the first one makes $\mathcal{M}$ into a module over the Lie K-algebra L in the sense of the Lie algebra theory, while the second map makes $\mathcal{M}$ into an A-module and additionally the following compatibility conditions hold

$$(am)x = m(ax) = a(mx) - x(a)m, \qquad a \in \mathrm{A}, m \in \mathcal{M} \text{ and } x \in L.$$

Definition 1.19. A *crossed module* $\partial \colon R \to L$ *of Lie–Rinehart algebras over* A (defined in [4]) consists of a Lie–Rinehart algebra L and a A-Lie algebra R together with an action of L on R and the Lie algebras homomorphism ∂ such that the following identities hold:

1. $\partial(x \circ r) = [x, \partial(r)]$,
2. $\partial(r') \circ r = [r', r]$,
3. $\partial(ar) = a\partial(r)$,
4. $\partial(r)(a) = 0$,

for all $a \in \mathrm{A}, r \in R$ and $x \in L$.

Example 1.20. We can see some examples of crossed modules of Lie–Rinehart algebras.

1. For any Lie–Rinehart homomorphism $f\colon L \to R$, the diagram $\mathsf{Ker}\, f \to L$ is a crossed module of Lie–Rinehart algebras.

2. If M is an ideal of L, the inclusion $M \hookrightarrow L$ is a crossed module where the action of L on M is given by the Lie bracket.

3. If R is a left Lie–Rinehart (A, L)-module with Lie bracket $[R, R] = 0$, the morphism $0\colon R \to L$ is a crossed module.

4. Let $\partial\colon R \to L$ be a central epimorphism $\big($i.e. $\mathsf{Ker}\,\partial \subset \mathsf{Z}(R)\big)$ from a Lie A-algebra R to a Lie–Rinehart algebra L which is also an A-Lie algebra. Then ∂ is a crossed module where the action from L to R is given by $x \circ r = [r', r]$, such that $\partial(r') = x$.

1.3 Lie–Rinehart Algebras and Poisson Algebras

Lie–Rinehart algebras are closely related to Poisson algebras. They both come from differential geometry and there are some similarities between them. In this section we will try to obtain Lie–Rinehart algebras from Poisson algebras and viceversa, and to see a geometrical point of view of this topic.

Definition 1.21. A *Poisson algebra* is a commutative K-algebra P equipped with a Lie K-algebra structure such that the following identity, called Leibniz rule, holds

$$[x, yz] = y[x, z] + [x, y]z,$$

where $x, y, z \in P$.

Example 1.22. We can see in [22] that if L is a Lie–Rinehart algebra over A, we can define a Poisson algebra $P = \mathrm{A} \oplus L$ with the two operations defined by

$$\begin{aligned} (a + x)(b + y) &:= ab + (ay + bx), \\ [a + x, b + y] &:= \big(x(b) - y(a)\big) + [x, y], \end{aligned}$$

where $a, b \in \mathrm{A}$ and $x, y \in L$. Conversely, any Poisson algebra P, whose underlying vector space can be split as $P = \mathrm{A} \oplus L$ and such that the two operations $\cdot$ and $[\ ,\ ]$ take values as follows:

$$\begin{aligned} \mathrm{A} \otimes \mathrm{A} &\xrightarrow{\cdot} \mathrm{A}, & \mathrm{A} \otimes \mathrm{A} &\xrightarrow{[\ ,\ ]} 0, \\ \mathrm{A} \otimes L &\xrightarrow{\cdot} L, & L \otimes \mathrm{A} &\xrightarrow{[\ ,\ ]} \mathrm{A}, \\ L \otimes L &\xrightarrow{\cdot} 0, & L \otimes L &\xrightarrow{[\ ,\ ]} L, \end{aligned}$$

defines a Lie–Rinehart algebra L over A. The two constructions are inverse to each other.

There are (at least) three Lie–Rinehart algebras related to any Poisson algebra P, which we will see them in the next examples.

Example 1.23. The first one is P itself considered as a P-module in an obvious way, where the action of P (as a Lie algebra) on P (as a commutative algebra) is given by the homomorphism $\mathsf{ad}\colon P \to \mathrm{Der}(P)$ given by $\mathsf{ad}(x) = [x, -] \in \mathrm{Der}(P)$.

There is a variant of this construction in the graded case. Let $P_* = \bigoplus_{n\geq 0} P_n$ be a commutative graded K-algebra in the sense of commutative algebra (i.e. no signs are involved) and assume P_* is equipped with a Poisson algebra structure such that the bracket has degree (-1). Thus $[-,-] : P_n \otimes P_m \to P_{n+m-1}$. Then P_1 is a Lie–Rinehart P_0-algebra, where the Lie algebra homomorphism $P_1 \to \mathrm{Der}(P_0)$ is given by $x_1 \mapsto [x_1, -]$, $[x_1, -](x_0) = [x_1, x_0]$, where $a_i \in P_i$, $i = 0, 1$.

Example 1.24. To see the second example, we establish

$$H^0_{Poiss}(P, P) := \{x \in P \mid [x, -] = 0\}.$$

Then $H^0_{Poiss}(P, P)$ contains the unit of P and is closed with respect to products, thus it is a subalgebra of P. A *Poisson derivation* of P is a linear map $D\colon P \to P$ which is a derivation simultaneously with respect to commutative and Lie algebra structures. We let $\mathsf{Der}_{Poiss}(P)$ be the collection of all Poisson derivations of P. It is closed with respect to the Lie bracket. Moreover, if $x \in H^0_{Poiss}(P, P)$ and $D \in \mathsf{Der}_{Poiss}(P)$ then $xD \in \mathsf{Der}_{Poiss}(P)$. It follows that $\mathsf{Der}_{Poiss}(P)$ is a Lie–Rinehart algebra over $H^0_{Poiss}(P, P)$.

To see the last one we need a definition first.

Definition 1.25. Given a commutative K-algebra B, we define the *Kähler differential* Ω^1_{B} as the kernel of the multiplication $\mathrm{B} \otimes \mathrm{B} \to \mathrm{B}$. We define the map $\mathrm{d}\colon \mathrm{B} \to \Omega^1_{\mathrm{B}}$ by $\mathrm{d}b = 1 \otimes b - b \otimes 1$, which is a derivation of B over K with values into Ω^1_{B}. It is clear from the definition that $\Omega^1_{\mathrm{B}} = \mathrm{Span}_{\mathrm{B}}\{\mathrm{d}b : b \in \mathrm{B}\}$, since the elements of Ω^1_{B} lie in the kernel of the multiplication map, if $\sum a_j \otimes b_j \in \Omega^1_{\mathrm{B}}$, then $\sum a_j b_j = 0$ and therefore

$$\sum a_j \otimes b_j = \sum (a_j \otimes b_j - a_j b_j \otimes 1) = \sum a_j \mathrm{d}b_j,$$

where $a_j, b_j \in \mathrm{B}$.

Example 1.26. Given a Poisson algebra P, we can extend by linearity the map $\mathrm{d}x \mapsto \mathsf{ad}(x)$ to get a morphism $\rho\colon \Omega^1_P \to \mathtt{Der}_K(P)$ uniquely defined by $\rho(\mathrm{d}x) = \mathsf{ad}(x)$ for all $x \in P$. Also given $\alpha \in \Omega^1_P$, we can define $\mathrm{d}\alpha$ through the formula

$$\mathrm{d}\alpha(D, D') = D\big(\alpha(D')\big) - D'\big(\alpha(D)\big) - \alpha([D, D']),$$

where $D, D' \in \mathtt{Der}_K(P)$.

In this way, Ω^1_P is a Lie–Rinehart algebra with anchor map ρ and Lie bracket

$$[x\mathrm{d}x', y\mathrm{d}y'] = x[x', y]\mathrm{d}y' + y[x, y']\mathrm{d}x' + xy\mathrm{d}[x', y'],$$

where $x, x', y, y' \in P$, and the bracket used on the left side of the identity is the bracket on the Poisson structure.

These examples give us some ways to associate Lie–Rinehart algebras from Poisson algebras. Now we will do the opposite of Example 1.26, find out when given a Lie–Rinehart algebra Ω^1_P we can form a Poisson structure on P. In addition, if Ω^1_P determines a Poisson structure on P, we will say that is Poisson type.

We can define the bracket on P in the following way

$$[x, y] = (\mathrm{d}y)\big(\rho(\mathrm{d}x)\big) = \big(\rho(\mathrm{d}x)\big)(y).$$

This bracket is K-linear and satisfies the Leibniz rule

$$[x, yz] = \big(\rho(\mathrm{d}x)\big)(yz) = y\rho(\mathrm{d}x)(z) + \rho(\mathrm{d}x)(y)z = y[x, z] + [x, y]z,$$

for all $x, y, z \in P$.

Theorem 1.27. *Let Ω^1_P a Lie–Rinehart algebra. Then there is a Poisson algebra structure on P such that $\rho(\mathrm{d}x) = \mathsf{ad}(x)$ for all $x \in P$ if and only if:*

1. $\mathrm{d}x\big(\rho(\mathrm{dd}y)\big) = -\mathrm{d}y\big(\rho(\mathrm{d}x)\big)$*, for all $x, y \in P$,*

2. *One of the following conditions holds:*

   (a) $\mathrm{d}\alpha = 0 = \mathrm{d}\beta$ *implies* $\mathrm{d}[\alpha, \beta] = 0$*, where $\alpha, \beta \in \Omega^1_P$.*

   (b) $[\mathrm{d}x, \mathrm{d}y] = \mathrm{d}\bigg(\mathrm{d}y\big(\rho(\mathrm{d}x)\big)\bigg)$ *for all $x, y \in P$.*

Under these conditions, the Lie bracket is reconstructed from ρ by the formula

$$[x\mathrm{d}x', y\mathrm{d}y'] = x[x', y]\mathrm{d}y' + y[x, y']\mathrm{d}x' + xy\mathrm{d}[x', y'].$$

Proof. The proof of this theorem is long and will be omitted but it can be found in [11]. □

When the anchor map ρ is injective, the property $[\mathrm{d}x, \mathrm{d}y] = \mathrm{d}[x, y]$ is trivial, since ρ is a Lie algebra morphism, so $\rho[\mathrm{d}x, \mathrm{d}y] = [\rho(\mathrm{d}x), \rho(\mathrm{d}y)] = \rho(\mathrm{d}[x, y])$. When ρ is not injective, this property can be true or not. We will see an example where is true.

Example 1.28. Let $K=\mathbb{R}$ and $P=\mathbb{R}[x^1,x^2,x^3]$. Then $\mathtt{Der}_K(P)=\operatorname{Span}\{\partial_1,\partial_2,\partial_3\}$ and $\Omega^1_P=\operatorname{Span}\{\mathrm{d}x^1,\mathrm{d}x^2,\mathrm{d}x^3\}$. We define the following bracket

$$\begin{aligned}[p_i\mathrm{d}x^i,q_j\mathrm{d}x^j]=&\big(p_1(\partial_2+\partial_3)+p_2(-\partial_1+\partial_3)+p_3(-\partial_1-\partial_2)\big)(q_i)\\ &-\big(q_1(\partial_2+\partial_3)+q_2(-\partial_1+\partial_3)+q_3(-\partial_1-\partial_2)\big)(p_i)\mathrm{d}x^i,\end{aligned}$$

and anchor map

$$\rho(p_i)\mathrm{d}x^i\mapsto-(p_2+p_3)\partial_1+(p_1-p_3)\partial_2+(p_1+p_2)\partial_3.$$

The matrix representation of ρ to the given basis in $\mathtt{Der}_K(P)$ and Ω^1_P is

$$\rho=\begin{pmatrix}0&1&1\\-1&0&1\\-1&-1&0\end{pmatrix}.$$

Therefore, ρ is in the condition of the first part of Theorem 1.27 and a long but straightforward computation, shows that Ω^1_P is a Lie–Rinehart algebra. Let us see that is of Poisson type checking the property $[\mathrm{d}x,\mathrm{d}y]=\mathrm{d}[x,y]$. For any $p,q\in P$,

$$\begin{aligned}\mathrm{d}[p,q]&=\mathrm{d}\big(\rho(\mathrm{d}p)(q)\big)=\mathrm{d}\big(\rho(\partial_ip\mathrm{d}x^i)(q)\big)\\&=\mathrm{d}(-(\partial_2p+\partial_3p)\partial_1q+(\partial_1p-\partial_3p)\partial_2q+(\partial_1q+\partial_2p)\partial_3q).\end{aligned}$$

The $\mathrm{d}x^1$ factor in the expansion of the expression (the other cases are similar) is

$$\begin{aligned}&-(\partial_2p+\partial_3p)\partial^2_{11}q+(\partial_1p-\partial_3p)\partial^2_{12}q+(\partial_1p+\partial_2p)\partial^2_{13}q\\&-(\partial^2_{12}p+\partial^2_{13}p)\partial_1q+(\partial^2_{11}p-\partial^2_{13})\partial_2q+(\partial^2_{11}p+\partial12^2p)\partial_3q.\end{aligned}$$

On the other hand, the bracket on the Lie–Rinehart algebra is

$$[\mathrm{d}p,\mathrm{d}q]=[\partial_ip\mathrm{d}x^i,\partial_jq\mathrm{d}x^j]=\rho(\mathrm{d}p)(\partial_kq)-\rho(\mathrm{d}q)(\partial_kp)\mathrm{d}x^k.$$

For $k=1$, we compute the coefficient of $\mathrm{d}x^1$,

$$\begin{aligned}&-(\partial_2p+\partial_3p)\partial^2_{11}q+(\partial_1p-\partial_3p)\partial^2_{12}q+(\partial_1p+\partial_2p)\partial^2_{13}q\\&-(\partial^2_{12}p+\partial^2_{13}p)\partial_1q+(\partial^2_{11}p-\partial^2_{13})\partial_2q+(\partial^2_{11}p+\partial12^2p)\partial_3q.\end{aligned}$$

which is the same as above. So Ω^1_P is a Lie–Rinehart algebra of Poisson type.

Chapter 2

Universal Enveloping Algebra

In this chapter we will see the definition and some characterizations of the Universal Enveloping Algebra of Lie algebras, and then we will extend them to Lie–Rinehart algebras. Then we will see some generalizations of Lie–Rinehart algebras and their correspondents universal enveloping algebras.

2.1 Universal Enveloping Algebra

Definition 2.1. Let L a Lie algebra, a *universal enveloping algebra of* L is an associative, unital algebra $\mathsf{U}\, L$ with the standard Lie bracket $[x, y] := xy - yx$ and a morphism $i\colon L \to \mathsf{U}\, L$, such that for any other associative, unital algebra B and a Lie morphism $\kappa\colon L \to \mathrm{B}$ there is a unique algebra homomorphism $f\colon \mathsf{U}\, L \to \mathrm{B}$ such that $f\big(i(x)\big) = \kappa(x)$.

It is easy to check that the universal enveloping algebra is unique up to isomorphism. In particular, the construction of $\mathsf{U}\, L$ is the K-algebra generated by the symbols $i(x)$ for each $x \in L$ satisfying the usual relations

$$i(kx) = ki(x),$$
$$i(x + y) = i(x) + i(y),$$

and the additional relation

$$i([x, y]) = i(x)i(y) - i(y)i(x),$$

for all $k \in K$ and $x, y \in L$.

Definition 2.2. A *filtration of an* K*-algebra* C is a sequence of K-submodules

$$C_0 \subset C_1 \subset \cdots C_n \subset \cdots \subset \bigcup_i C_i = C,$$

such that $C_iC_j \subset C_{i+j}$. The *associated graded algebra* of a filtration is defined by $\mathsf{gr}_*(C) = \bigoplus_{n\geq 0} C_n/C_{n-1}$ with multiplication $\mathsf{gr}_i\mathsf{gr}_j \to \mathsf{gr}_{i+j}$. Note that the universal enveloping algebra $\mathsf{U}\,L$ defines an algebra filtration where C_i are the K-submodules generated by $i(x_1)\cdots i(x_i)$.

Now we move to the case of Lie–Rinehart algebras.

Definition 2.3. Let L a Lie–Rinehart algebra over A. Let $\mathsf{U}(\mathrm{A}\oplus L)$ the universal enveloping algebra of the Lie algebra $(\mathrm{A}\oplus L)$, with bracket

$$[(a,x),(b,y)] = (x(b) - y(a), [x,y]).$$

If $i\colon \mathrm{A}\oplus L \to \mathsf{U}(\mathrm{A}\oplus L)$ is the canonical inclusion, we write $\bar{\mathsf{U}}(\mathrm{A}\oplus L)$ for the subalgebra generated by $i(\mathrm{A}\oplus L)$. The *universal enveloping algebra of* L is the quotient

$$\mathsf{U}_{\mathrm{A}}\,L = \bar{\mathsf{U}}(\mathrm{A}\oplus L)/I,$$

where I is the two-sided ideal in $\bar{\mathsf{U}}(\mathrm{A}\oplus L)$ generated by the elements $i(a,0)\cdot i(b,x) - i(ab,ax)$ for all $a,b\in \mathrm{A}$ and $x\in L$.

In particular, another way to see the universal enveloping algebra of L, is that $\mathsf{U}_{\mathrm{A}}\,L$ is the algebra generated by the symbols $j(a)$ for each $a\in\mathrm{A}$ and $i(x)$ for each $x\in L$, satisfying the following relations

$$\begin{aligned} j(1) &= 1,\\ j(ab) &= j(a)j(b),\\ i(ax) &= j(a)i(x),\\ i([x,y]) &= i(x)i(y) - i(y)i(x),\\ i(x)j(a) &= j(a)i(x) + j\big(x(a)\big). \end{aligned}$$

The universal enveloping algebra $\mathsf{U}_{\mathrm{A}}\,L$ is characterized by the following universal property: if B is any K-algebra, $\kappa_{\mathrm{A}}\colon \mathrm{A}\to\mathrm{B}$ is a homomorphism of K-algebras and $\kappa_L\colon L\to\mathrm{B}$ is a homomorphism of Lie algebras such that $\kappa_{\mathrm{A}}(a)\kappa_L(x) = \kappa_L(ax)$ and $[\kappa_L(x),\kappa_{\mathrm{A}}(a)] = \kappa_A\big(x(a)\big)$ for any $a\in\mathrm{A}$ and $x\in L$, then there exists a unique homomorphism of algebras $f:\mathsf{U}_{\mathrm{A}}\,L\to\mathrm{B}$ such that $f\big(i(x)\big) = \kappa_L(x)$ for all $x\in L$ and $f\big(j(a)\big) = \kappa_{\mathrm{A}}(a)$ for all $a\in\mathrm{A}$.

In particular, the universal property of $\mathsf{U}_{\mathrm{A}}\,L$ implies that there exists a unique representation

$$\varrho\colon\ \mathsf{U}_{\mathrm{A}}\,L \to \mathrm{End}_K(\mathrm{A})$$

such that $\varrho\circ i = \rho$ and $\varrho\circ j$ is the canonical representation given by the multiplication in A.

Example 2.4. If L is a Lie A-algebra as in Example 1.3, the universal enveloping algebra $\mathsf{U}_{\mathrm{A}}\, L$ is the classical universal enveloping algebra $\mathsf{U}(L)$ of L.

Example 2.5. Let $K = \mathbb{R}$, M a manifold, $\mathrm{A} = \mathcal{C}^\infty(\mathrm{M})$ and $L = \mathtt{Der}_K\big(\mathcal{C}^\infty(\mathrm{M})\big)$. This way, L is a Lie–Rinehart algebra with anchor map the identity map, and the universal enveloping algebra $\mathsf{U}_{\mathrm{A}}\, L$ is the ring of global differential operators on M.

It is clear that there is a one-to-one correspondence between left (A, L)-modules and left $\mathsf{U}_{\mathrm{A}}\, L$-modules. In particular, the obvious (A, L)-module structure on A induces on A a left $\mathsf{U}_{\mathrm{A}}\, L$-module structure given by

$$\mu\colon\ \mathsf{U}_{\mathrm{A}}\, L \otimes \mathrm{A} \to \mathrm{A}, \qquad \mu(x \otimes a) = x(a).$$

Let V_n be the A-submodule spanned on all products $i(x_1)\cdots i(x_k)$, where $k \leq n$. Then

$$0 \subset \mathrm{A} = V_0 \subset V_1 \subset \cdots \subset V_n \subset \cdots \subset \mathsf{U}_{\mathrm{A}}\, L,$$

defines an algebra filtration on $\mathsf{U}_{\mathrm{A}}\, L$. It is clear that $\mathsf{U}_{\mathrm{A}}\, L = \cup_{n\geq 0} V_n$. It follows from the fourth relation of the definition that the associated graded object $\mathsf{gr}_*(V)$ is a commutative A-algebra. In other words $\mathsf{U}_{\mathrm{A}}\, L$ is an almost commutative algebra in the following sense.

Definition 2.6. An *almost commutative algebra* is an associative K-algebra C together with a filtration

$$0 \subset \mathrm{A} = C_0 \subset C_1 \subset \cdots \subset C_n \subset \cdots \subset C = \bigcup_{n\geq 0} C_n$$

$C_n C_m \subset C_{n+m}$ and such that the associated graded object $\mathsf{gr}_*(C) = \bigoplus_{n\geq 0} C_n/C_{n-1}$ is a commutative A-algebra.

Remark 2.7. If C is an almost commutative algebra, then there is a well-defined bracket

$$[-,-]\colon \mathsf{gr}_n(C) \otimes \mathsf{gr}_m(C) \longrightarrow \mathsf{gr}_{n+m-1}(C)$$

which is given as follows. Let $a \in \mathsf{gr}_n(C)$ and $b \in \mathsf{gr}_m(C)$ and $\hat{a} \in C_n$ and $\hat{b} \in C_m$ representing a and b respectively. Since $\mathsf{gr}_*(C)$ is a commutative algebra it follows that $\hat{a}\hat{b} - \hat{b}\hat{a} \in C_{n+m-1}$ and the corresponding class in $\mathsf{gr}_{n+m-1}(C)$ is $[a, b]$. It is also well known that in this way we obtain a Poisson algebra structure on $\mathsf{gr}_*(C)$. Since the bracket is of degree (-1) by Example 1.23, $L = \mathsf{gr}_1(C)$ is a Lie–Rinehart algebra over $\mathrm{A} = \mathsf{gr}_0(C)$. Moreover the exact sequence

$$0 \to \mathrm{A} \to C_1 \to L \to 0,$$

is an abelian extension of Lie–Rinehart algebras (see below Definition 3.25).

Proposition 2.8. *The correspondence assigning C_1 to the almost commutative algebra C, defines a functor $LR\colon \mathtt{AComm}_{\mathrm{A}} \to \mathtt{LR}_{\mathrm{AK}}$.*

Proof. Let $f\colon C \to D \in \mathtt{AComm}_{\mathrm{A}}$. Since f preserves the filtration, $f(C_1) \subset D_1$. Furthermore, $f(ax) = f(a)f(x) = af(x)$, for any $a \in C_0 = D_0$ and $x \in C_1$, and $f([x,y]) = f(xy - yx) = f(x)f(y) - f(y)f(x) = [f(x), f(y)]$, for $x, y \in C_1$. Hence the restriction of f to C_1, which we shall call $LR(f)$, is a morphism of K-Lie algebras and of A-modules such that the following diagram commutes in $\mathtt{Lie}_{\mathrm{K}}$,

$$\begin{tikzcd} C_1 \arrow[rr, "LR(f)"] \arrow[dr, "{[\circ,-]}"'] & & D_1 \arrow[dl, "{[\circ,-]}"] \\ & \mathtt{Der}_K(\mathrm{A}) & \end{tikzcd}$$

Thus, $LR(f) \in \mathtt{LR}_{\mathrm{AK}}$.

On the other hand, it is clear that $LR(1_{C_1}) = 1_{C_1}$ and the following diagram commutes in K-mod,

$$\begin{tikzcd} C \arrow[r, "f"] & D \arrow[r, "g"] & E \\ C_1 \arrow[u, hook, "i_C"] \arrow[r, "LR(f)"] & D_1 \arrow[u, hook, "i_D"'] \arrow[r, "LR(g)"] & E_1 \arrow[u, hook, "i_E"'] \end{tikzcd}$$

Hence LR is functorial. $\square$

Proposition 2.9. *The functor LR is right adjoint to the universal enveloping functor $\mathsf{U}_{\mathrm{A}}\colon \mathtt{LR}_{\mathrm{AK}} \to \mathtt{AComm}_{\mathrm{A}}$.*

Proof. Let $\Phi\colon \mathtt{AComm}_{\mathrm{A}}(\mathsf{U}_{\mathrm{A}}\,\mathrm{L}, \mathrm{C}) \to \mathtt{LR}_{\mathrm{AK}}(\mathrm{L}, \mathtt{LR}\mathrm{C})$ be the map given as follows. Since $\mathsf{U}_{\mathrm{A}}\,L$ is generated as a K-algebra by L and A, a morphism $f\colon\ \mathsf{U}_{\mathrm{A}}\,L \to C$ is completely determined by its restriction to L and A. Since $f(a) = a$ for every $a \in \mathrm{A}$, and $f(L) \subset f\big((\mathsf{U}_{\mathrm{A}}\,L)_1\big) \subset C_1$, it follows that the restriction of f to L, $\Phi f\colon L \to C_1 = \mathtt{LR}\mathrm{C}$ is a monomorphism of Lie–Rinehart algebras.

Let $g\colon L \to C_1 \in \mathtt{LR}_{\mathrm{AK}}$. We build up the morphism $\widetilde{g}\colon\ \mathsf{U}_{\mathrm{A}}\,L \to C$ by $\widetilde{g}(ax_1 \cdots x_m) := ag(x_1)\cdots g(x_m) \in C$. It is straightforward to see that $\widetilde{g} \in \mathtt{AComm}_{\mathrm{A}}$ and $\Phi\widetilde{g} = g$. Hence Φ is bijective, and U_{A} and $\mathtt{LR}$ form an adjoint pair. $\square$

2.2 PBW Theorem

In this section we will give the classical result theorem of Poincaré-Birkhoff-Witt in Lie algebras, and we will prove the analogue for Lie–Rinehart algebras first proved by Rinehart in [28].

Definition 2.10. The *tensor algebra* of an K-module $\mathcal{N}$, is the graded K-algebra

$$T(\mathcal{N}) = \bigoplus_{n\geq 0} T^n(\mathcal{N}),$$

where $T^0(\mathcal{N}) = K$ and $T^n(\mathcal{N}) = \mathcal{N}\otimes \overbrace{\cdots}^{n} \otimes\mathcal{N}$.

The universal enveloping algebra of a Lie algebra L, is just $T(L)/I$ where I is the ideal generated by the elements $x\otimes y - y\otimes x - [x,y]$, where $x, y \in L$.

Definition 2.11. The *symmetric algebra* of $\mathcal{N}$ is

$$\mathrm{Sym}(\mathcal{N}) = T(\mathcal{N})/I,$$

where I is the ideal generated by the symbols $m\otimes n - n\otimes m$, for $m, n \in \mathcal{N}$.

Lemma 2.12. *If $G(\mathsf{U}\, L)$ is the universal enveloping algebra with the graded structure, the canonical epimorphism $T(L) \to \mathsf{U}\, L$ factorizes into an epimorphism $\mathrm{Sym}(L) \to G(\mathsf{U}\, L)$.*

Proof. It is easy to see since the elements of the form $x\otimes y - y\otimes x \in T(L)_2$ are sent to $[x,y] \in (\mathsf{U}\, L)_1$. ☐

Theorem 2.13 (PBW Theorem)**.** *If L is a Lie algebra and it is K-free, then the canonical epimorphism,*

$$\mathrm{Sym}(L) \to G(\mathsf{U}\, L),$$

is an isomorphism.

Corollary 2.14. *If L is free as an K-module with basis $\{x_i, i \in I\}$, the set*

$$\{x_{i_1}^{s_i} x_{i_2}^{s_2} \cdots x_{i_r}^{s_r} : i_1 < i_2 < \cdots < i_r \text{ and } s_i \geq 0\},$$

form an K-basis of $\mathsf{U}\, L$. In particular $i\colon L \to \mathsf{U}\, L$ is injective.

These are classical results of Lie algebras. Now we will move into Lie–Rinehart algebras. We will need some previous lemmas to prove the results.

Lemma 2.15. *Let A be a ring, $\mathcal{M}_i$ be a right A-module for every i in some index set, and let $\mathcal{N}$ be a projective left A-module. Then, the natural homomorphism*

$$(\prod_i \mathcal{M}_i) \otimes_{\mathrm{A}} \mathcal{N} \to \prod_i (\mathcal{M}_i \otimes_{\mathrm{A}} B),$$

is injective.

Lemma 2.16. *Let* A *be a commutative ring. The natural map*

$$\mathrm{A} \to \prod_{\mathfrak{p} \in \mathrm{Max(A)}} \mathrm{A}_{\mathfrak{p}},$$

where Max(A) *are the maximal ideals of* A *and* $\mathrm{A}_{\mathfrak{p}}$ *is the corresponding local ring, is injective.*

Proof. The proofs of these lemmas can be found in [28]. □

Definition 2.17. Let L a Lie–Rinehart algebra over A and let $\mathcal{M}$ a Lie module for $\mathrm{A} \oplus L$. We say that $\mathcal{M}$ is an A-*regular* L-*module* if

$$a(zm) = (az)m,$$

for all $a \in \mathrm{A}$, $z \in \mathrm{A} \oplus L$ and $m \in \mathcal{M}$.

The canonical map $\mathrm{A} \oplus L \to \mathsf{U}_{\mathrm{A}}\, L$ endows any $\mathsf{U}_{\mathrm{A}}\, L$-module with the structure of an A-regular L-module. Thus we have a one-to-one correspondence between $\mathsf{U}_{\mathrm{A}}\, L$-modules and A-regular L-modules. In particular, A has a natural structure as an A-regular L-module, and the representation of A thus obtained is faithful. Hence the map $\mathrm{A} \to \mathsf{U}_{\mathrm{A}}\, L$ is injective and we will identify A with its image in $\mathsf{U}_{\mathrm{A}}\, L$.

If $\mathcal{M}$ and $\mathcal{N}$ are $\mathsf{U}_{\mathrm{A}}\, L$-modules, we can define an A-regular L-module on $\mathcal{M} \otimes \mathcal{N}$ such that

$$a(m \otimes n) = (am) \otimes n = m \otimes (an),$$
$$x(m \otimes n) = (xm) \otimes n + m \otimes (xn),$$

for $a \in \mathrm{A}$, $x \in L$, $m \in \mathcal{M}$ and $n \in \mathcal{N}$.

Lemma 2.18. *Let* $\mathcal{N}$ *a left* $\mathsf{U}_{\mathrm{A}}\, L$*-module. The* $\mathsf{U}_{\mathrm{A}}\, L$*-module* $\mathcal{N} \otimes \mathsf{U}_{\mathrm{A}}\, L$ *as defined above is isomorphic to* $\mathsf{U}_{\mathrm{A}}\, L \otimes \mathcal{N}$ *with the usual left* $\mathsf{U}_{\mathrm{A}}\, L$*-module structure.*

Proof. The proof can be read in [13]. □

Given the graded algebra structure on $\mathsf{U}_{\mathrm{A}}\, L$, there is a canonical map $T(L) \to \mathsf{U}_{\mathrm{A}}\, L$, and it factorizes into a map $\mathrm{Sym}(L) \to \mathrm{G}(\mathsf{U}_{\mathrm{A}}\, L)$. This leads to the analogue of the Poincaré-Birkhoff-Witt theorem of Lie algebras.

Theorem 2.19 (PBW Theorem)**.** *If* L *is a Lie–Rinehart algebra over* A*, and is* A*-projective, then the canonical* A*-module epimorphism,*

$$\mathrm{Sym}(L) \to \mathrm{G}(\mathsf{U}_{\mathrm{A}}\, L)$$

where $\mathrm{G}(\mathsf{U}_{\mathrm{A}}\, L)$ *is* U_{A} *seen as a graded* A*-algebra, is an* A*-algebra isomorphism.*

Proof. First of all, we will prove the theorem assuming that L is A-free. Let $\{x_i\}$ be an ordered A-basis of L. We denote by X_i the elements x_i considered as elements of $\mathrm{Sym}(L)$, and we denote the image of x_i in $\mathsf{U}_{\mathrm{A}}\, L$ by the inclusion of L by $\bar{x}_i$. If I is a sequence $i_1 \leq \cdots \leq i_n$, let $X_I = X_{i_1} \cdots X_{i_n}$. If I is the empty sequence, let $X_I = 1$. We write $j \leq I$ in case either $j \leq i_1$ or I is empty. The main part of the proof is to define an A-regular L-module structure on $\mathrm{Sym}(L)$ such that, if $j \leq I$, $x_j X_I = X_j X_I$. Once we have this structure, we obtain an $\mathsf{U}_{\mathrm{A}}\, L$-module structure for $\mathrm{Sym}(L)$ because the one-to-one correspondence, and this structure, will have the property that for any ordered sequence I, $(\bar{x}_{i_1}, \ldots, \bar{x}_{i_n}) \cdot 1 = X_I$. Since the X_I form an A-basis for $\mathrm{Sym}(L)$ it suffices to prove the theorem.

Let $\mathrm{Sym}^p(L)$ denote the homogeneous component of degree p of $\mathrm{Sym}(L)$ and let $Q_p = \sum_{q=0}^{p} \mathrm{Sym}^p(L)$. We will define by induction a K-bilinear map $L \times \mathrm{Sym}(L) \to \mathrm{Sym}(L)$, denoted by $(x, Y) \mapsto xY$ by defining its restriction $L \times Q_p \to Q_{p+1}$ for each p, subject to the following conditions:

$$x_j X_I = X_j X_I \qquad \text{if } j \leq I \text{ and } X_I \in Q_p, \tag{2.1}$$

$$x(x'Y) = x'(xY) + [x, x']Y \qquad \text{if } x, x' \in L \text{ and } Y \in Q_{p-1}, \tag{2.2}$$

$$x_j X_I - X_J X_I \in Q_q \qquad \text{if } X_I \in Q_q \text{ and } q \leq p, \tag{2.3}$$

$$(ax)(bY) = a(b(xY) + x(b)Y) \qquad \text{if } a, b \in \mathrm{A}, x \in L \text{ and } Y \in Q_p. \tag{2.4}$$

For $p = 0$, we define $xa = ax + x(a)$, satisfying conditions (2.1) through (2.4).

Now suppose we have already defined an action $L \times Q_{p-1} \to Q_p$ satisfying the conditions corresponding to (2.1) through (2.4). In order to extend this, we first define the action by the elements x_i mapping $\mathrm{Sym}^p(L)$ into Q_{p+1}. We may assume inductively that we have defined this action for all x_j such that $j < i$. Let $X_I \in \mathrm{Sym}^p(L)$, if $i \leq I$, we define $x_i X_I = X_i X_I$. If not, then $I = (j, J)$ with $j < i$, and by induction hypothesis we can define $x_i X_I = x_j(x_i X_J) + [x_i, x_j]X_J$. Now we define the action by x_i on all of $\mathrm{Sym}^p(L)$ by defining $x_i(aX_I) = a(x_i X_I) + x_i(a)X_I$ for $a \in \mathrm{A}$ and extending by K-linearity. Thus we have defined the action by the elements x_i. To define the action on $\mathrm{Sym}^p(L)$ by an arbitrary element of L, define $(ax_i)Y = a(x_i Y)$ for $a \in \mathrm{A}$ and $Y \in \mathrm{Sym}^p(L)$, and we extend it by K-linearity. Conditions (2.1), (2.3) and (2.4) are clearly satisfied. We just have to see the verification of condition (2.3),

$$x_i(x_j X_H) = x_j(x_i X_H) + [x_i, x_j]X_H.$$

If $i > j$, with $j \leq H$, then $(j, H) = J$, and using the definition of the action, the condition follows immediately. If $j > i$ and $i \leq H$, since the Lie bracket is skew-symmetric it is the same as before. In addition, if $i = j$ condition (2.3) trivially holds. So let us consider the case when neither $i \leq H$ nor $j \leq H$. Then H has positive length and $H = (g, G)$ where $g \leq G$, $g < i$ and $g < j$. By the inductive assumption,

$$x_j(X_H) = x_j(x_g X_G) = x_g(x_j X_G) + [x_j, x_g]X_G = x_g(x_j X_G) + x_g w + [x_j, x_g]X_G,$$

where $w = x_j X_G - X_j X_G \in Q_{p-1}$. Applying x_i to both sides we have

$$x_i(x_j X_H) = x_i\big(x_g(x_j X_G)\big) + x_i(x_g w) + x_i([x_j, x_g]X_L).$$

Since $g \leq (j, G)$, condition (2.3) may be applied, and after some computation we get

$$x_i(x_j X_H) = x_g\big(x_i(x_j X_G)\big) + [x_i, x_g](x_j X_G) + [x_j, x_g](x_i X_G) + \big[x_i, [x_j, x_g]\big]X_G.$$

Our assumptions on i and j were symmetric, so the previous identity holds if we change of position i and j. Subtracting and applying condition (2.3) again, the final result is

$$x_i(x_j X_H) - x_j(x_i X_H) = [x_i, x_j]X_H.$$

Therefore we have an action by elements of L on $\mathrm{Sym}(L)$. We use this to define an action of $\mathrm{A} \oplus L$ on $\mathrm{Sym}(L)$ in the obvious way. Using (2.2) and (2.4) it is easy to see that this endows $\mathrm{Sym}(L)$ with the structure of an A-regular L-module, and we have proven the theorem when L is A-free.

Now we assume only that L is A-projective. Let $\mathfrak{p}$ any prime ideal of A. If D is any K-derivation of A, the formula $D_{\mathfrak{p}}(a/b) = \big(bD(a) - aD(b)\big)/b^2$ extends D to a K-derivation of $\mathrm{A}_{\mathfrak{p}}$. Thus L is represented on $\mathrm{A}_{\mathfrak{p}}$. Let $L_{\mathfrak{p}} = \mathrm{A}_{\mathfrak{p}} \otimes_{\mathrm{A}} L$ with the natural $\mathrm{A}_{\mathfrak{p}}$-module structure. We can define a bracket on $L_{\mathfrak{p}}$

$$[a \otimes x, b \otimes y] = ab \otimes [x, y] + ax(b) \otimes y - by(a) \otimes x,$$

where $a, b \in A_{\mathfrak{p}}$ and $x, y \in L$. It is a straightforward computation to check that this is a Lie bracket. In addition, the elements of $L_{\mathfrak{p}}$ act as derivations of $\mathrm{A}_{\mathfrak{p}}$ in the natural way, so $L_{\mathfrak{p}}$ becomes a Lie–Rinehart algebra over $\mathrm{A}_{\mathfrak{p}}$.

Since L is A-projective, so is $\mathrm{Sym}(L)$, and hence the monomorphism of Lemma 2.16 and the injection of Lemma 2.15, there is a monomorphism

$$\mathrm{Sym}(L) = \mathrm{A} \otimes_{\mathrm{A}} \mathrm{Sym}(L) \to (\prod \mathrm{A}_{\mathfrak{p}}) \otimes_{\mathrm{A}} \mathrm{Sym}(L) \to \prod \big(\mathrm{A}_{\mathfrak{p}} \otimes_{\mathrm{A}} \mathrm{Sym}(L)\big) = \prod \mathrm{Sym}(L_{\mathfrak{p}}),$$

where the product is taken over all the maximal ideals of A. The natural A-module and Lie algebra homomorphism $\mathrm{A} \oplus L \to \mathrm{A}_{\mathfrak{p}} \oplus L_{\mathfrak{p}}$ defines an A-algebra homomorphism $\mathsf{U}_{\mathrm{A}}\, L \to \mathsf{U}_{\mathrm{A}}\, L_{\mathfrak{p}}$. Therefore we have a map $\mathsf{U}_{\mathrm{A}}\, L \to \prod(\mathsf{U}_{\mathrm{A}}\, L_{\mathfrak{p}})$. This map is compatible with the filtration of $\mathsf{U}_{\mathrm{A}}\, L$ and $\mathsf{U}_{\mathrm{A}}\, L_{\mathfrak{p}}$ so we obtain a map $\mathrm{G}(\mathsf{U}_{\mathrm{A}}\, L) \to \prod \mathrm{G}(\mathsf{U}_{\mathrm{A}}\, L_{\mathfrak{p}})$. Since L is A-projective, $L_{\mathfrak{p}}$ is $\mathrm{A}_{\mathfrak{p}}$-projective. Hence, since $\mathrm{A}_{\mathfrak{p}}$ is a local ring, $L_{\mathfrak{p}}$ is $\mathrm{A}_{\mathfrak{p}}$-free (this can be found in [18]. By the first part of the proof, the map $\mathrm{Sym}(L_\mathfrak{p}) \to \mathrm{G}(\mathsf{U}_\mathrm{A}\, L_\mathfrak{p})$ is therefore an isomorphism. Hence we have the commutative and exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
 & \mathrm{Sym}(L) \arrow[r] \arrow[d] & \mathrm{G}(\mathsf{U}_\mathrm{A}\, L) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \prod \mathrm{Sym}(L_\mathfrak{p}) \arrow[r] & \prod \mathrm{G}(\mathsf{U}_\mathrm{A}\, L_\mathfrak{p}) \arrow[r] & 0
\end{tikzcd}$$

where we deduce that the arrow of the top row is injective, completing the proof. □

In the particular case of Example 1.3, this theorem is the usual Poincaré-Birkhoff-Witt theorem for Lie algebras.

Corollary 2.20. *If L is a Lie–Rinehart algebra over* A*, projective as an* A*-module, the map*

$$i\colon L \to \mathsf{U}_\mathrm{A}\, L,$$

is injective.

Corollary 2.21. *If L is free as an* A*-module with basis $\{x_i, i \in I\}$, the set*

$$\{x_{i_1}^{s_i} x_{i_2}^{s_2} \cdots x_{i_r}^{s_r} : i_1 < i_2 < \cdots < i_r \text{ and } s_i \geq 0\},$$

form an A*-basis of $\mathsf{U}_\mathrm{A}\, L$.*

2.3 Left Hopf Algebroid Structure

An important field of study in Lie algebras, is the fact that the universal enveloping algebra is a Hopf algebra. In the case of Lie–Rinehart algebras, the universal enveloping algebra does not need to be a Hopf algebra, but it is a left Hopf algebroid. In this section we will introduce the notions of Hopf algebras and left Hopf algebroids and we will give the universal enveloping algebras each of the correspondent structures.

Definition 2.22. It is well known that a K-algebra A is a triple $(\mathrm{A}, m_\mathrm{A}, \eta)$, where $m_\mathrm{A}\colon \mathrm{A}\otimes \mathrm{A} \to \mathrm{A}$, $\eta\colon K \to \mathrm{A}$ and the associativity and the unitality properties follow, i.e. $m_\mathrm{A}(m_\mathrm{A} \otimes \mathrm{id}_\mathrm{A}) = m_\mathrm{A}(\mathrm{id}_\mathrm{A} \otimes m_\mathrm{A})$ and $m_\mathrm{A}(\eta \otimes \mathrm{id}_\mathrm{A}) = m_\mathrm{A}(\mathrm{id}_\mathrm{A} \otimes \eta) = \mathrm{id}_\mathrm{A}$. Analogously, one can say that A is a *K-coalgebra* if it is a triple $(\mathrm{A}, \Delta, \epsilon)$, such that $\Delta\colon \mathrm{A} \to \mathrm{A} \otimes \mathrm{A}$ and $\epsilon\colon \mathrm{A} \to K$ satisfying the coassociativity and the counitality properties

$$(\Delta \otimes \mathrm{id}_\mathrm{A})\Delta = (\mathrm{id}_\mathrm{A} \otimes \Delta)\Delta,$$
$$(\mathrm{id}_\mathrm{A} \otimes \epsilon)\Delta = (\epsilon \otimes \mathrm{id}_\mathrm{A})\Delta = \mathrm{id}_\mathrm{A}.$$

Definition 2.23. A K-*bialgebra* is a quintuple $(\mathrm{A}, m_{\mathrm{A}}, \eta, \Delta, \epsilon)$, where $(\mathrm{A}, m_{\mathrm{A}}, \eta)$ is a K-algebra and $(\mathrm{A}, \Delta, \epsilon)$ is a K-coalgebra verifying the following equivalent conditions

(i) The maps m_{A} and η are morphisms of K-coalgebras.

(ii) The maps Δ and ϵ are morphisms of K-algebras.

Definition 2.24. Let now $(H, m_H, \eta, \Delta, \epsilon)$ be a K-bialgebra. An endomorphism $S\colon H \to H$ is called an *antipode* for H if

$$m_H(S \otimes \mathrm{id}_H)\Delta = m_H(\mathrm{id}_H \otimes S)\Delta = \eta\epsilon.$$

A *Hopf algebra* is a K-bialgebra with an antipode.

Example 2.25. The universal enveloping algebra $\mathsf{U}\, L$ of a Lie algebra L is a Hopf algebra with structure maps defined on generators in the following way

$$\begin{aligned}\Delta(x) &= x \otimes 1 + 1 \otimes x,\\ \epsilon(x) &= 0,\\ S(x) &= -x,\end{aligned}$$

for all $x \in L$.

Now we move to K-algebroids and Lie–Rinehart algebras.

Definition 2.26. Let A be a K-algebra, the *opposite* ring A^{op} is the same structure as A but the product ab in A^{op} is defined by ba in A. We define the *enveloping algebra of* A as $\mathrm{A}^{\mathrm{e}}{:=}\mathrm{A} \otimes \mathrm{A}^{\mathrm{op}}$.

Definition 2.27. An A-*ring* U is a triple (U, m_U, η) where U is an A^{e}-module, $m_U : U \otimes_{\mathrm{A}} U \to U$, $u \otimes v \to uv$ and $\eta\colon \mathrm{A} \to U$ are (A, A)-bimodule maps such that

$$\begin{aligned}m_U(m_U \otimes \mathrm{id}_U) &= m_U(\mathrm{id}_U \otimes m_U),\\ m_U(\eta \otimes \mathrm{id}_U) &= m_U(\mathrm{id}_U \otimes \eta).\end{aligned}$$

These properties are the associativity and the unitality. It can be seen in [1] that the A-rings U correspond bijectively to K-algebra homomorphisms

$$\eta\colon \mathrm{A} \to U.$$

Definition 2.28. Dually to the notion of an A-ring is the concept of an A-coring. This is, an A-coring C is a triple (C, Δ, ϵ) where C is an (A, A)-bimodule (with left and right actions L_{A} and R_{A}) and $\Delta\colon C \to C \otimes_{\mathrm{A}} C$, $\epsilon\colon C \to \mathrm{A}$ are (A,A)-bimodule homomorphisms such that

$$(\Delta \otimes \mathrm{id}_C)\Delta = (\mathrm{id}_C \otimes \Delta)\Delta,$$
$$L_{\mathrm{A}}(\epsilon \otimes \mathrm{id}_C)\Delta = R_{\mathrm{A}}(\mathrm{id}_C \otimes \epsilon)\Delta = \mathrm{id}_C.$$

These properties are the coassociativity and the counitality.

Given an A^{e}-ring U, described by a K-algebra map $\eta\colon \mathrm{A}^{\mathrm{e}} \to U$, we can consider the restrictions

$$s := \eta(- \otimes 1_{\mathrm{A}})\colon \mathrm{A} \to U,$$
$$t := \eta(1_{\mathrm{A}} \otimes -)\colon \mathrm{A}^{\mathrm{op}} \to U.$$

We will call these maps *source* and *target* map of the A^{e}-ring. In this way, and A^{e}-ring may be equally given by such a triple (U, s, t). Using the left A^{e}-module structure $(a \otimes \tilde{a}, u) \mapsto \eta(a \otimes \tilde{a})u$ on U, one considers

$$\begin{aligned} U \otimes_{\mathrm{A}} U :&= U \otimes U/\mathrm{span}\{\eta(1 \otimes a)u \otimes u' - u \otimes \eta(a \otimes 1)u' : a \in \mathrm{A} \text{ and } u, u' \in U\} \\ &= U \otimes U/\mathrm{span}\{t(a)u \otimes u' - u \otimes s(a)u' : a \in \mathrm{A} \text{ and } u, u' \in U\}. \end{aligned}$$

Definition 2.29. We will call the *left Takeuchi product* of the A^{e}-ring U with itself to the K-submodule of $U \otimes_{\mathrm{A}} U$

$$U \times_{\mathrm{A}} U := \{\sum_i u_i \otimes_{\mathrm{A}} u'_i \in U \otimes_{\mathrm{A}} U : \sum_i u_i t(a) \otimes_{\mathrm{A}} u'_i = \sum u_i \otimes_{\mathrm{A}} u'_i s(a) \text{ for all } a \in \mathrm{A}\}.$$

Definition 2.30. A *left* A*-bialgebroid* is a K-module U that carries simultaneously the structure of an A^{e}-ring (U, s, t) and a A-coring (U, Δ, ϵ), subject to the following compatibility axioms:

1. We have an (A,A)-bimodule structure described by

$$a \triangleright u \triangleleft \tilde{a} := \eta(a \otimes \tilde{a})u = s(a)t(\tilde{a})u$$

for $a, \tilde{a} \in \mathrm{A}$ and $u \in U$. We will refer to this structure by writing ${}_{\triangleright}U_{\triangleleft}$.

2. Considering the bimodule ${}_{\triangleright}U_{\triangleleft}$, the coproduct Δ is a K-algebra morphism taking values in $U \times_{\mathrm{A}} U$.

3. For all $a, \tilde{a} \in \mathrm{A}, u, u' \in U$, the counit ϵ has the properties

$$\epsilon(s(a)t(\tilde{a}u) = a\epsilon(u)\tilde{a},$$
$$\epsilon(uu') = \epsilon\big(us(\epsilon u')\big) = \epsilon\big(ut(\epsilon u')\big).$$

Observe that a left bialgebroid carries in total *four* A-module structure, because one also has

$$a \blacktriangleright u \blacktriangleleft \tilde{a} := u\eta(\tilde{a} \otimes a) = us(\tilde{a})t(a)$$

and we will denote this situation by ${}_{\blacktriangleright}U_{\blacktriangleleft}$.

Definition 2.31. Let U be a left A-bialgebroid, we define the *Galois map* of U by

$$\beta \colon {}_{\blacktriangleright}U \otimes_{\mathrm{A}^{\mathrm{op}}} U_{\triangleleft} \to U_{\triangleleft} \otimes_{\mathrm{A}} {}_{\triangleright}U, \qquad u \otimes_{\mathrm{A}^{\mathrm{op}}} v \mapsto u_{(1)} \otimes_{\mathrm{A}} u_{(2)}v,$$

where $u_{(1)}$ and $u_{(2)}$ are the first and second components of $\Delta(u)$, and

$${}_{\blacktriangleright}U \otimes_{\mathrm{A}^{\mathrm{op}}} U_{\triangleleft} = U \otimes U/\mathrm{span}\{a \blacktriangleright u \otimes v - u \otimes v \triangleleft a : a \in \mathrm{A}, u, v \in U\}.$$

In this way, we will say that U is a *left Hopf algebroid* if β is a bijection.

Now we are in conditions to see the canonical left Hopf algebroid structure on $\mathsf{U}_{\mathrm{A}}\, L$ of a Lie–Rinehart algebra L. The source and targets maps, are given by the inclusion of A in $\mathsf{U}_{\mathrm{A}}\, L$. In this way,

$$a \triangleright u \triangleleft \tilde{a} := au\tilde{a}$$

We write $\mathsf{U}_{\mathrm{A}}\, L \otimes^{\mathsf{U}} \mathsf{U}_{\mathrm{A}}\, L := \mathsf{U}_{\mathrm{A}}\, L_{\triangleleft} \otimes_{\mathrm{A}} {}_{\triangleright}\, \mathsf{U}_{\mathrm{A}}\, L$, and $\mathsf{U}_{\mathrm{A}}\, L \times^{\mathsf{U}} \mathsf{U}_{\mathrm{A}}\, L := \mathsf{U}_{\mathrm{A}}\, L \times_{\mathrm{A}} \mathsf{U}_{\mathrm{A}}\, L$ for the Takeuchi product. We can define the coproduct on generators as

$$\Delta(x) = 1 \otimes^{\mathsf{U}} x + x \otimes^{\mathsf{U}} 1,$$
$$\Delta(a) = a \otimes^{\mathsf{U}} 1,$$

which maps $a \in \mathrm{A}$ and $x \in L$ into $\mathsf{U}_{\mathrm{A}}\, L \times^{\mathsf{U}} \mathsf{U}_{\mathrm{A}}\, L$ and can be extended by the universal property to a coproduct

$$\Delta \colon \mathsf{U}_{\mathrm{A}}\, L \to \mathsf{U}_{\mathrm{A}}\, L \times^{\mathsf{U}} \mathsf{U}_{\mathrm{A}}\, L \subset \mathsf{U}_{\mathrm{A}}\, L \otimes^{\mathsf{U}} \mathsf{U}_{\mathrm{A}}\, L.$$

The counit is similarly given by extension of the anchor map α to $\mathsf{U}_{\mathrm{A}}\, L$, more precisely, by

$$\epsilon(u) \mapsto \alpha(u)(1_{\mathrm{A}}),$$

so in particular $\epsilon(a) = a$ if $a \in \mathrm{A}$ and $\epsilon(x) = 0$ if $x \in L$.

We write now $\mathsf{U}_{\mathrm{A}}\, L \otimes^{\mathrm{rl}} \mathsf{U}_{\mathrm{A}}\, L := {}_{\blacktriangleright}\, \mathsf{U}_{\mathrm{A}}\, L \otimes_{\mathrm{A}^{\mathrm{op}}} \mathsf{U}_{\mathrm{A}}\, L_{\triangleleft}$, and $\mathsf{U}_{\mathrm{A}}\, L \times^{\mathrm{rl}} \mathsf{U}_{\mathrm{A}}\, L := \mathsf{U}_{\mathrm{A}}\, L \times_{\mathrm{A}^{\mathrm{op}}} \mathsf{U}_{\mathrm{A}}\, L$.

We define on generators in $\mathsf{U}_{\mathrm{A}}\, L \otimes^{\mathrm{rl}} \mathsf{U}_{\mathrm{A}}\, L$

$$a_{+} \otimes^{\mathrm{rl}} a_{-} := a \otimes^{\mathrm{rl}} 1$$
$$x_{+} \otimes^{\mathrm{rl}} x_{-} := x \otimes^{\mathrm{rl}} 1 - 1 \otimes^{\mathrm{rl}} x.$$

In this way, we define a map $\beta^{-1}(-\otimes^{\mathsf{U}}1)\colon U \to \mathsf{U}_{\mathrm{A}}\, L\otimes^{\mathrm{rl}}\mathsf{U}_{\mathrm{A}}\, L$. This map stay in $\mathsf{U}_{\mathrm{A}}\, L\times^{\mathrm{rl}}\mathsf{U}_{\mathrm{A}}\, L$ which is an algebra through the product of $\mathsf{U}_{\mathrm{A}}\, L$ in the first and its opposite in the second tensor factor. By universality we obtain a map $\mathsf{U}_{\mathrm{A}} \to \mathsf{U}_{\mathrm{A}}\, L \times^{\mathrm{rl}} \mathsf{U}_{\mathrm{A}}\, L \subset \mathsf{U}_{\mathrm{A}}\, L \otimes^{\mathrm{rl}} \mathsf{U}_{\mathrm{A}}\, L$, and then β^{-1} is defined by

$$\beta^{-1}(u \otimes^{\mathsf{U}} v) = u_{+} \otimes^{\mathrm{rl}} u_{-}v.$$

Since the map β^{-1} is well defined, $\mathsf{U}_{\mathrm{A}}\, L$ is a left Hopf algebroid.

Conversely, certain left bialgebroids give rise to Lie–Rinehart algebras:

Proposition 2.32. *Let $(U, \mathrm{A}, s, t, \Delta, \epsilon)$ be a left algebroid with* A *commutative and $s \equiv t$. Therefore, the module $P^{\ell}U = \{u \in U : \Delta(u) = u \otimes 1 + 1 \otimes u\}$ is a Lie–Rinehart algebra over* A*.*

Proof. We just have two remaining A-module structures on U, denoted by $au := s(a)u$ and $ua := us(a)$. The coproduct is a map $\Delta\colon U \to U \otimes^{\mathsf{U}} U$, where we use again the same notation as above. The natural Lie algebra structure is $[u, u'] = uu' - u'u$, which is closed in $P^{\ell}U$. We have that $\Delta(au) = au \otimes^{\mathsf{U}} 1 + 1 \otimes^{\mathsf{U}} au$ for $u \in P^{\ell}U$, which is an A-submodule. The anchor map is given by the Lie algebra action

$$\alpha(u)(a) = \epsilon(ua) =: u(a).$$

To see the last property, let $a, b \in \mathrm{A}$ and $u, u' \in U$,

$$\begin{aligned}
([u, au'])(b) &= \epsilon\big(u(au'(b))\big) - au'\big(u(b)\big)\\
&= u(a)u'(b) - a([u, u'])(b)\\
&= (u(a)u')(b) - (a[u, u'])(b).
\end{aligned}$$

Since b was arbitrary, the proof is complete. □

2.4 Lie–Rinehart Superalgebras

As in many algebraic and geometric structures, we have a definition of a Lie–Rinehart superalgebra. We will give some examples of Lie–Rinehart superalgebras and we will also give the definition of the universal enveloping superalgebra.

Definition 2.33. A *superalgebra* is a $\mathbb{Z}_2$-graded algebra, that is, a direct sum $\mathrm{A} = \mathrm{A}_0 \oplus \mathrm{A}_1$ of two subspaces A_0 and A_1 that satisfy $\mathrm{A}_i\mathrm{A}_j \subset \mathrm{A}_{i+j}$ for all $i, j \in \mathbb{Z}_2$. We have a map $|\cdot|\colon (A_0 \sqcup A_1)\backslash\{0\} \to \mathbb{Z}_2$ which sends the elements of A_0 to 0, and the elements of A_1 to 1, and we can extend this map by linearity to the whole space A. We call the elements of $\mathrm{A}_0 \sqcup \mathrm{A}_1$ *homogeneous.* We say that a superalgebra A is *commutative* if

$$yx = (-1)^{|x||y|}xy, \qquad \text{for all } x, y \text{ homogeneous in A.}$$

A *Lie superalgebra* is a superalgebra whose product satisfies the conditions of super skew-symmetry and super Jacobi identity,

$$[x,y] = -(-1)^{|x||y|}[y,x],$$
$$\big[x,[y,z]\big] = \big[[x,y],z\big] + (-1)^{|x||y|}\big[y,[x,z]\big],$$

for all x, y homogeneous in A. If L is a Lie superalgebra, an endomorphism $D \in \mathrm{End}(L)_s$ is called a *derivation of degree* s, for $s \in \mathbb{Z}_2$, if

$$D(xy) = D(x)y + (-1)^{s|x|}xD(b), \qquad \text{for all } x, y \text{ homogeneous in A.}$$

One verifies that $\mathtt{Der}_K(L) = \mathtt{Der}_K(L)_0 \oplus \mathtt{Der}_K(L)_1$ is a Lie superalgebra.

Definition 2.34. Let A be a supercommutative, associative, unital K-superalgebra, and let L be a Lie superalgebra over K which is also an A-module. Assume that we are given $\alpha_L\colon L \to \mathtt{Der}_K(\mathrm{A})$ a morphism of Lie superalgebras (i.e. a morphism of Lie algebras such that conserve the subspaces) and of A-modules. We say that L is a *Lie–Rinehart superalgebra over* A if for all $a \in \mathrm{A}$ and $x, y \in L$, we have

$$[x, ay] = a[x,y](-1)^{|a||x|} + \alpha_L(x)(a)y.$$

Example 2.35. Let A be a supercommutative, associative, unital superalgebra and let $\mathfrak{g}$ be a Lie superalgebra. Assume that there is a Lie superalgebra morphism $\alpha\colon \mathfrak{g} \to \mathtt{Der}_K(\mathrm{A})$. The Lie superalgebra $L = \mathrm{A} \otimes \mathfrak{g}$ with Lie bracket

$$[a \otimes x, b \otimes y] = (-1)^{|x||b|}ab \otimes [x,y] + a\alpha(x)(b) \otimes y - b\alpha(y)a \otimes x(-1)^{(|a|+|x|)(|b|+|y|)},$$

where $a, b \in \mathrm{A}$ and $x, y \in \mathfrak{g}$, is a Lie–Rinehart superalgebra extending the map α to an A-module morphism $\alpha_L\colon L \to \mathtt{Der}_K(\mathrm{A})$.

We can define the universal enveloping superalgebra of a Lie–Rinehart superalgebra in the same way as we did for Lie–Rinehart algebras.

Definition 2.36. Let L a Lie–Rinehart superalgebra over A. The *universal enveloping superalgebra*, denoted by $\mathsf{U}_{\mathrm{A}}\, L$, is the K-superalgebra generated by the symbols $i(x)$ for each $x \in L$ and $j(a)$ for each $a \in \mathrm{A}$, satisfying

$$j(1) = 1,$$
$$j(ab) = j(a)j(b),$$
$$i(ax) = j(a)i(x),$$
$$i([x,y]) = i(x)i(y) - (-1)^{|x||y|}i(y)i(x),$$
$$i(x)j(a) = (-1)^{|a||x|}j(a)i(x) + j\big(x(a)\big).$$

As in the case of Lie–Rinehart algebras, it can be defined a filtration and the PBW theorem is also true as we can see in [6].

Example 2.37. Let $K = \mathbb{R}$ and M a paracompact smooth supermanifold ([20]) over K. Doing like in Example 2.5, with $\mathrm{A} = \mathcal{C}^\infty(\mathrm{M})$, the superalgebra $L = \mathtt{Der}_K\big(\mathcal{C}^\infty(\mathrm{M})\big)$ is a Lie–Rinehart superalgebra with anchor map the identity, and the universal enveloping superalgebra is the superalgebra of differential operators over M.

Example 2.38. Let L_A be a Lie–Rinehart algebra over A and let L_B be a Lie–Rinehart algebra over B. We define

$$L_{\mathrm{A}\otimes\mathrm{B}} = \mathrm{B}\otimes L_\mathrm{A} \oplus \mathrm{A}\otimes L_\mathrm{B}.$$

Then, $L_{\mathrm{A}\otimes\mathrm{B}}$ is an $\mathrm{A}\otimes\mathrm{B}$-module. We define now an $\mathrm{A}\otimes\mathrm{B}$-module morphism $\alpha_{\mathrm{A}\otimes\mathrm{B}}\colon L_{\mathrm{A}\otimes\mathrm{B}} \to \mathtt{Der}_K(\mathrm{A}\otimes\mathrm{B})$ by

$$\begin{aligned}
\alpha_{\mathrm{A}\otimes\mathrm{B}}(x)(a\otimes b) &= \alpha_{L_\mathrm{A}}(x)(a)\otimes b,\\
\alpha_{\mathrm{A}\otimes\mathrm{B}}(y)(a\otimes b) &= a\otimes \alpha_{L_\mathrm{B}}(y)(-1)^{|a||y|},
\end{aligned}$$

for all $a \in \mathrm{A}$, $b \in \mathrm{B}$, $x \in L_\mathrm{A}$ and $y \in L_\mathrm{B}$. We also define a Lie bracket in $L_{\mathrm{A}\otimes\mathrm{B}}$ extending the particular Lie brackets and $[L_\mathrm{A}, L_\mathrm{B}] = 0$. In this way, $L_{\mathrm{A}\otimes\mathrm{B}}$ is a Lie–Rinehart superalgebra over $\mathrm{A}\otimes\mathrm{B}$ with anchor map $\alpha_{\mathrm{A}\otimes\mathrm{B}}$.

2.5 Restricted Lie–Rinehart Algebras

In many cases when we study Lie algebras over a field of prime characteristic we are led to consider a richer structure than an ordinary Lie algebra. In this section, K will be a field of characteristic p prime.

Definition 2.39. A *restricted Lie algebra* $(L, (-)^{[p_L]})$ over a field K is a Lie algebra L together with a map $(-)^{[p_L]}\colon L \to L$, called the p-map such that the following relations hold

$$\begin{aligned}
kx^{[p_L]} &= k^p x^{[p_L]},\\
[x, y^{[p_L]}] &= \big[[[x, \underbrace{y], y], \cdots, y}_{p}\big],\\
(x+y)^{[p_L]} &= x^{[p_L]} + y^{[p_L]} + \sum_{i=1}^{p-1} s_i(x,y),
\end{aligned}$$

where $is_i(x,y)$ is the coefficient of λ^{i-1} in $\mathsf{ad}^{p-1}_{\lambda x+y}(x)$. The motivation of this definitions can be found in [17].

Let K be a field of characteristic $p \neq 0$, and A a unital K-algebra. If $D \in \mathtt{Der}_K(A)$, it follows the formula

$$D^p(ab) = \sum_{i=0}^{p} \binom{p}{i} D^i(a) D^{p-i}(b),$$

for all $a, b \in \mathrm{A}$. Since the characteristic of K is p we get

$$D^p(ab) = aD^p(b) + D^p(a)b,$$

which means that $D^p \in \mathtt{Der}_K(A)$, so $(\mathtt{Der}_K(A), (-)^p)$ is a restricted Lie algebra. Moreover, we can get the relation

$$(aD)^p = a^p D^p + (aD)^{p-1}(a)D,$$

so we are naturally led to the following definition.

Definition 2.40. A *restricted Lie–Rinehart algebra over* A is a restricted Lie algebra over K, $(L, (-)^{[p_L]})$, where L is a Lie–Rinehart algebra over A, the anchor map also conserves the operation $(-)^{[p_L]}$ (i.e. is a restricted Lie homomorphism), and the following relation holds:

$$(ax)^{[p_L]} = a^p x^{[p_L]} + (ax)^{p-1}(a)x,$$

for all $a \in \mathrm{A}$ and $x \in L$.

Example 2.41. As we have seen, $\mathtt{Der}_K(A)$ is a restricted Lie–Rinehart algebra.

Example 2.42. Any restricted Lie algebra over K is a restricted Lie–Rinehart algebra over K with trivial anchor map.

Example 2.43. If $\mathfrak{g}$ is a restricted Lie algebra with a homomorphism of restricted Lie algebras $\gamma\colon \mathfrak{g} \to \mathtt{Der}_K(\mathrm{A})$, then the transformation Lie–Rinehart algebra $\mathrm{A} \otimes \mathfrak{g}$ can be endowed with a restricted Lie–Rinehart algebra structure, with bracket

$$[a \otimes g, a' \otimes g'] := aa' \otimes [g, g'] + a\gamma(g)(a') \otimes g' - a'\gamma(g')(a) \otimes g,$$

anchor map $\alpha_L\colon L \to \mathtt{Der}_K(\mathrm{A})$,

$$\alpha_L(a \otimes g)(a') = a\gamma(g)(a')$$

and p-map

$$(a \otimes g)^{[p_L]} = a^p \otimes g^{[p_L]} - \big(a\gamma(g)\big)^{p-1}(a) \otimes g,$$

where $a, a' \in \mathrm{A}$ and $g, g' \in \mathfrak{g}$.

Example 2.44. Let $F : K$ be a purely inseparable field extension of exponent 1. Then, there is a one-to-one correspondence between intermediate fields and restricted Lie–Rinehart subalgebras of $\mathtt{Der}_K(F)$, seen as a restricted Lie–Rinehart algebra over F.

Since a restricted Lie–Rinehart algebra is in particular a Lie–Rinehart algebra, one can construct the universal enveloping algebra in the way of this chapter, and in [9] can be found a variant of Corollary 2.21.

Theorem 2.45. *Let L a restricted Lie–Rinehart algebra over* A *and free as an* A*-module. If $\{x_i, i \in I\}$ is an* A*-basis of L, then the set*

$$\{z_{i_1}^{h_i} z_{i_2}^{h_2} \cdots z_{i_r}^{h_r} x_{i_1}^{k_i} x_{i_2}^{k_2} \cdots x_{i_r}^{k_r}\}$$

where $i_1 < i_2 < \cdots i_r$, $h_i \geq 0$, $0 \leq k_i < p$ and $z_i = x_i^p - x_i^{[p_L]}$, is an A*-basis of L.*

Moreover, one can define an analogue of the universal enveloping algebra on restricted Lie–Rinehart algebras.

Definition 2.46. Let $(L, (-)^{[p_L]})$ be a restricted Lie–Rinehart algebra, we define the *restricted universal enveloping algebra* as the quotient $\mathsf{U}_{\mathrm{A}}\, L/I$, where I is generated by the elements $\{x_i^p - x_i^{[p_L]}\}$. This algebra, follows the same universal property as the universal enveloping algebra of Lie–Rinehart algebras, but with restricted morphisms.

Chapter 3

Homology and Cohomology

3.1 Lie Algebras Homology

In this section we will introduce the Lie algebra homology and cohomology theory in order to extend it to Lie–Rinehart algebras in the next section. The proofs of these classical results can be found in [29].

Definition 3.1. Given a K-module $\mathcal{M}$, the *free Lie algebra on* $\mathcal{M}$ is a Lie algebra $\mathrm{F}(\mathcal{M})$, containing $\mathcal{M}$ as a submodule, which satisfies the following universal property: Every K-module map $\mathcal{M} \to L$ into a Lie algebra, extends uniquely to a Lie algebra map $\mathrm{F}(\mathcal{M}) \to L$. In other words, as a functor F is left adjoint to the forgetful functor from Lie algebras to modules

$$\mathsf{Hom}_{K-\mathrm{mod}}(\mathcal{M}, L) \cong \mathsf{Hom}_{\mathrm{Lie}}(\mathrm{F}(\mathcal{M}), L).$$

Example 3.2. The free Lie algebra of the free K-module with one generator x, is the 1-dimensional abelian Lie algebra K. The free Lie algebra of the free K-module with two generators $\{x, y\}$ is the free K-module having an infinite basis of monomials

$$x, y, [x, y], \big[x, [x, y]\big], \big[y, [x, y]\big], \big[x, [x, [x, y]]\big], \big[x, [y, [x, y]]\big], \big[y, [y, [x, y]]\big], \ldots$$

Definition 3.3. The *exterior algebra* of an K-module $\mathcal{N}$, is the graded K-algebra

$$\Lambda_K(\mathcal{N}) := T(\mathcal{N})/I$$

where $T(\mathcal{N})$ is the tensor algebra and I is the ideal generated by the symbols $n \otimes n$ for all $n \in \mathcal{N}$. We denote by $\Lambda_K^p(\mathcal{N})$ his object of degree p.

Definition 3.4. If L is a K-Lie algebra and $\mathcal{M}$ is a Lie module over L, we define the chain complex

$$C_n^{\mathrm{Lie}}(L, \mathcal{M}) := \mathcal{M} \otimes \Lambda_K^n(L), \qquad n \geq 0,$$

with boundary map

$$\partial\colon C_n^{\mathrm{Lie}}(L,\mathcal{M}) \longrightarrow C_{n-1}^{\mathrm{Lie}}(L,\mathcal{M}),$$

defined by

$$\begin{aligned}\partial\big(m\otimes(x_1,\dots,x_n)\big) &= \sum_{i=1}^{n}(-1)^{(i-1)}\, x_i m\otimes(x_1,\dots,\hat{x_i},\dots,x_n)\\ &\qquad + \sum_{j<k}(-1)^{j+k}\, m\otimes([x_j,x_k],x_1,\dots,\hat{x_j},\dots,\hat{x_k},\dots,x_n),\end{aligned}$$

where $x_1,\dots,x_n\in L, m\in\mathcal{M}$.

In this way, the *Lie homology* is defined by

$$H_n^{\mathrm{Lie}}(L,\mathcal{M}) = H_n\big(C_n^{\mathrm{Lie}}(L,\mathcal{M})\big),\qquad n\geq 0.$$

Definition 3.5. If L is a Lie algebra and $\mathcal{M}$ is a Lie module over L, we define now the cochain complex

$$C^n_{\mathrm{Lie}}(L,\mathcal{M}) := \mathsf{Hom}_K(\Lambda^n_K L,\mathcal{M}),\qquad n\geq 0,$$

with coboundary map

$$\delta\colon C^{n-1}_{\mathrm{Lie}}(L,\mathcal{M}) \longrightarrow C^n_{\mathrm{Lie}}(L,\mathcal{M}),$$

defined by

$$\begin{aligned}(\delta f)(x_1,\dots,x_n) &= \sum_{i=1}^{n}(-1)^{(i-1)}\, x_i\big(f(x_1,\dots,\hat{x_i},\dots,x_n)\big)\\ &\qquad + \sum_{j<k}(-1)^{j+k}\, f([x_j,x_k],x_1,\dots,\hat{x_j},\dots,\hat{x_k},\dots,x_n),\end{aligned}$$

where $x_1,\dots,x_n\in L, m\in\mathcal{M}, f\in C^{n-1}(L,\mathcal{M})$.

In this way, the *Lie cohomology* is defined by

$$H^n_{\mathrm{Lie}}(L,\mathcal{M}) = H^n\big(C^n_{\mathrm{Lie}}(L,\mathcal{M})\big),\qquad n\geq 0.$$

Another way to see the Lie homology, is by the derived functors. If $\mathcal{M}$ is a L-module which is K-projective, it can be seen in [29] that

$$\begin{aligned}H_n^{\mathrm{Lie}}(L,\mathcal{M}) &\cong \mathsf{Tor}_n^{\mathsf{U}\,L}(K,\mathcal{M}),\\ H^n_{\mathrm{Lie}}(L,\mathcal{M}) &\cong \mathsf{Ext}^n_{\mathsf{U}\,L}(K,\mathcal{M}).\end{aligned}$$

where $\mathsf{U}\,L$ denotes the universal enveloping algebra of the Lie algebra L.

Proposition 3.6. *Let $L = \mathrm{F}(X)$ be the free Lie algebra on a free K-module generated by a set X. Then*

$$H_n^{\mathrm{Lie}}(L, \mathcal{M}) = 0, \qquad n > 1,$$
$$H^n_{\mathrm{Lie}}(L, \mathcal{M}) = 0, \qquad n > 1.$$

In addition, if $\mathcal{M} = K$, then $H_0^{\mathrm{Lie}}(L, K) = H^0_{\mathrm{Lie}}(L, K) = 0$ and

$$H_n^{\mathrm{Lie}}(L, \mathcal{M}) = \bigoplus_{x \in X} K,$$
$$H^n_{\mathrm{Lie}}(L, \mathcal{M}) = \prod_{x \in X} K.$$

Now we will describe the low homology of Lie algebras. By definition,

$$H_0^{\mathrm{Lie}}(L, \mathcal{M}) = \mathcal{M}_L = \frac{\mathcal{M}}{\mathcal{M} \circ L},$$

is the module of *coinvariants* of $\mathcal{M}$, where $\mathcal{M} \circ L$ means the K-submodule of $\mathcal{M}$ generated by mx, $x \in L, m \in \mathcal{M}$. In the same way, the *invariant K-submodule* of $\mathcal{M}$ is

$$H^0_{\mathrm{Lie}}(L, \mathcal{M}) = \mathcal{M}^L = \{m \in \mathcal{M} \mid xm = 0 \text{ for all } x \in L\}.$$

Proposition 3.7. *If $\mathcal{M}$ is any trivial L-module (i.e. $xm = 0$ for all $m \in \mathcal{M}$ and $x \in L$), then $H_1^{\mathrm{Lie}}(L, \mathcal{M}) \cong L^{\mathrm{ab}} \otimes \mathcal{M}$.*

Definition 3.8. If $\mathcal{M}$ is a L-module, a *derivation* from L into $\mathcal{M}$ is a K-linear map $D \colon L \to \mathcal{M}$ such that the Leibniz formula holds

$$D([x, y]) = x(Dy) - y(Dx).$$

The set of all derivations from L into $\mathcal{M}$ is denoted by $\mathsf{Der}_K(L, \mathcal{M})$ and it is a K-submodule of $\mathsf{Hom}_K(L, \mathcal{M})$.

An *inner derivation* is a derivation of $\mathsf{Der}_K(L, \mathcal{M})$ which is defined by $D_m(x) = xm$. They form K-submodule $\mathsf{Der}_K(L, \mathcal{M})$ and it is denoted by $\mathsf{IDer}_K(L, \mathcal{M})$.

Proposition 3.9. *If $\mathcal{M}$ is a L-module,*

$$H^1_{\mathrm{Lie}}(L, \mathcal{M}) \cong \frac{\mathsf{Der}_K(L, \mathcal{M})}{\mathsf{IDer}_K(L, \mathcal{M})}.$$

Corollary 3.10. *If $\mathcal{M}$ is a trivial L-module,*

$$H^1_{\mathrm{Lie}}(L, \mathcal{M}) \cong \mathsf{Der}_K(L, \mathcal{M}) \cong \mathsf{Hom}_{\mathrm{Lie}}(L, \mathcal{M}) \cong \mathsf{Hom}_K(L^{\mathrm{ab}}, \mathcal{M}).$$

Definition 3.11. Let L be a Lie algebra and let $\mathcal{M}$ be a L-module considered as an abelian Lie algebra. An *abelian extension* of L by $\mathcal{M}$ is an short exact sequence

$$0 \longrightarrow \mathcal{M} \xrightarrow{i} L' \xrightarrow{\partial} L \longrightarrow 0,$$

where L' is a Lie algebra, i is an K-linear map.

Proposition 3.12. *Let L a Lie algebra and $\mathcal{M}$ a L-module. There is a one-to-one correspondence between $H^2_{\mathrm{Lie}}(L, \mathcal{M})$ and abelian extensions of L by $\mathcal{M}$. In addition, the extension*

$$0 \longrightarrow \mathcal{M} \longrightarrow L \rtimes \mathcal{M} \longrightarrow L \longrightarrow 0$$

represents the $0 \in H^2_{\mathrm{Lie}}(L, \mathcal{M})$.

Definition 3.13. A *central extension* of a Lie algebra L is a short exact sequence

$$0 \longrightarrow I \xrightarrow{i} E \xrightarrow{p} L \longrightarrow 0,$$

where we identify I with $\mathsf{Ker}\, p$, and $\mathsf{Ker}\, P \subset \mathsf{Z}(E) = \{x \in E : [x, y] = 0 \text{ for all } y \in E\}$. In this way, an extension of L is an surjective Lie homomorphism $p\colon E \to L$ such that $\mathsf{Ker}\, p \subset \mathsf{Z}(E)$. If $p\colon E \to L$ and $p'\colon E' \to L$, a *homomorphism* from p to p' is a commutative diagram of the form

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & L & \end{tikzcd}$$

A central extension $\mathfrak{u}\colon \mathcal{L} \to L$ is called *universal* if there exists a unique homomorphism from $\mathfrak{u}$ to any other central extension of L. If L has a central extension, if follows immediately that is unique.

Theorem 3.14. *A Lie algebra L has a universal central extension if and only if L is perfect (i.e. $L = [L, L]$). In this case, if $\mathfrak{u}\colon \mathcal{L} \to L$ is the central extension, we have that $\mathsf{Ker}\, \mathfrak{u} = H_2^{\mathrm{Lie}}(L, K)$.*

3.2 Lie–Rinehart Algebras Homology

Now we will generalize the previous section to Lie–Rinehart algebras.

Definition 3.15. We recover from [5] the definition of the free Lie–Rinehart algebra. Let ${}_{\mathrm{A}}\mathrm{mod}/\mathtt{Der}_{\mathrm{K}}(\mathrm{A})$ be the category of K-linear maps $\psi\colon \mathcal{M} \to \mathtt{Der}_K(\mathrm{A})$ where $\mathcal{M}$ is a K-module. A morphism $\psi \to \psi_1$ in ${}_{\mathrm{A}}\mathrm{mod}/\mathtt{Der}_{\mathrm{K}}(\mathrm{A})$ is a K-linear map $f\colon \mathcal{M} \to \mathcal{N}$ such that $\psi = \psi_1 f$. We have the functor

$$U\colon \mathtt{LR}_{\mathrm{AK}} \to {}_{\mathrm{A}}\mathrm{mod}/\mathtt{Der}_{\mathrm{K}}(\mathrm{A}),$$

which assigns $\alpha_L \colon L \to \mathtt{Der}_K(\mathrm{A})$ to a Lie–Rinehart algebra L. We construct the functor

$$\mathrm{FR}\colon {}_{\mathrm{A}}\mathrm{mod}/\mathtt{Der}_{\mathrm{K}}(\mathrm{A}) \to \mathtt{LR}_{\mathrm{AK}},$$

as follows. Let $\psi\colon \mathcal{M} \to \mathtt{Der}_K(\mathrm{A})$ be a K-linear map. We let $\mathrm{F}(\mathcal{M})$ be the free Lie K-algebra generated by $\mathcal{M}$. Then we have the unique Lie K-algebra homomorphism $\mathrm{F}(\mathcal{M}) \to \mathtt{Der}_K(\mathrm{A})$ which extends the map ψ, which is still denoted by ψ. Now we can apply the construction from Example 1.8 to get a Lie–Rinehart algebra structure on $\mathrm{A} \otimes \mathrm{F}(\mathcal{M})$. We let $\mathrm{FR}(\psi)$ be this particular Lie–Rinehart algebra and we call it *the free Lie–Rinehart algebra generated by ψ*. In this way, we obtain the functor FR, which is left adjoint to U,

$$\mathsf{Hom}_{\mathrm{A}\,\mathrm{mod}/\mathtt{Der}_{\mathrm{K}}(\mathrm{A})}(\psi, \alpha_L) \cong \mathsf{Hom}_{\mathtt{LR}_{\mathrm{AK}}}(\mathrm{FR}(\psi), L).$$

Definition 3.16. If L is a Lie–Rinehart algebra over A and $\mathcal{M}$ is a right (A, L)-module over, we define the chain complex

$$C_n^{\mathrm{A}}(L, \mathcal{M}) := \mathcal{M} \otimes_{\mathrm{A}} \Lambda_{\mathrm{A}}^n(L), \qquad n \geq 0,$$

with boundary map

$$\partial\colon C_n^{\mathrm{A}}(L, \mathcal{M}) \longrightarrow C_{n-1}^{\mathrm{A}}(L, \mathcal{M}),$$

defined by

$$\begin{aligned}\partial\big(m \otimes_{\mathrm{A}} (x_1, \dots, x_n)\big) = \sum_{i=1}^{n} &(-1)^{(i-1)}\, m x_i \otimes_{\mathrm{A}} (x_1, \dots, \hat{x_i}, \dots, x_n) \\ &+ \sum_{j<k} (-1)^{j+k}\, m \otimes_{\mathrm{A}} ([x_j, x_k], x_1, \dots, \hat{x_j}, \dots, \hat{x_k}, \dots, x_n),\end{aligned}$$

where $x_1, \dots, x_n \in L, m \in \mathcal{M}$.

In this way, the *Lie–Rinehart homology* is defined by

$$H_n^{\mathrm{Rin}}(L, \mathcal{M}) = H_n\big(C_n^{\mathrm{A}}(L, \mathcal{M})\big), \qquad n \geq 0.$$

Definition 3.17. If L is a Lie–Rinehart algebra and $\mathcal{M}$ is a left (A, L)-module, we define now the cochain complex

$$C_{\mathrm{A}}^n(L, \mathcal{M}) := \mathsf{Hom}_{\mathrm{A}}(\Lambda_{\mathrm{A}}^n L, \mathcal{M}), \qquad n \geq 0,$$

with coboundary map

$$\delta\colon C_{\mathrm{A}}^{n-1}(L, \mathcal{M}) \longrightarrow C_{\mathrm{A}}^n(L, \mathcal{M}),$$

defined by

$$\begin{aligned}(\delta f)(x_1, \dots, x_n) = \sum_{i=1}^{n} &(-1)^{(i-1)}\, x_i\big(f(x_1, \dots, \hat{x_i}, \dots, x_n)\big) \\ &+ \sum_{j<k} (-1)^{j+k}\, f([x_j, x_k], x_1, \dots, \hat{x_j}, \dots, \hat{x_k}, \dots, x_n),\end{aligned}$$

where $x_1, \dots, x_n \in L, m \in \mathcal{M}, f \in C^{n-1}_{\mathrm{A}}(L, \mathcal{M})$.

In this way, the *Lie–Rinehart cohomology* is defined by

$$H^n_{\mathrm{Rin}}(L, \mathcal{M}) = H^n\big(C^n_{\mathrm{A}}(L, \mathcal{M})\big), \qquad n \geq 0.$$

Another way to see the Lie cohomology, is by the derived functor Ext. If $\mathcal{M}$ is a left (A, L)-module which is A-projective, it can be seen in [14] that

$$H^n_{\mathrm{Rin}}(L, \mathcal{M}) \cong \mathsf{Ext}^n_{\mathsf{U}_{\mathrm{A}}\, L}(\mathrm{A}, \mathcal{M}).$$

where $\mathsf{U}_{\mathrm{A}}\, L$ denotes the universal enveloping algebra of the Lie–Rinehart algebra L.

Example 3.18. If $\mathrm{A} = K$, then we recover the classical definition of the Lie algebra homology and cohomology. For a general A by forgetting the A-module structure one obtains the canonical homomorphisms

$$H^{\mathrm{Lie}}_n(L, \mathcal{M}) \to H^{\mathrm{Rin}}_n(L, \mathcal{M}), \qquad H^n_{\mathrm{Rin}}(L, \mathcal{M}) \to H^n_{\mathrm{Lie}}(L, \mathcal{M}).$$

On the other hand if A is a smooth commutative algebra, then $H^n_{\mathrm{Rin}}(\mathtt{Der}_K(\mathrm{A}), \mathrm{A})$ is isomorphic to the de Rham cohomology of A (this results can be found in [14] and [28]).

Lemma 3.19. *Let $\mathfrak{g}$ be a K-Lie algebra acting on a commutative algebra* A *by derivations and let L be the transformation Lie–Rinehart algebra of* $(\mathfrak{g}, \mathrm{A})$ *defined in Example 1.8. Then for any Lie–Rinehart* (A, L)*-module* $\mathcal{M}$ *we have the canonical isomorphisms of complexes* $C^{\mathrm{A}}_n(L, \mathcal{M}) \cong C^{\mathrm{Lie}}_n(\mathfrak{g}, \mathcal{M})$*,* $C^n_{\mathrm{A}}(L, \mathcal{M}) \cong C^n_{\mathrm{Lie}}(\mathfrak{g}, \mathcal{M})$ *and in particular the isomorphisms*

$$H^{\mathrm{Rin}}_n(L, \mathcal{M}) \cong H^{\mathrm{Lie}}_n(\mathfrak{g}, \mathcal{M}),$$
$$H^n_{\mathrm{Rin}}(L, \mathcal{M}) \cong H^n_{\mathrm{Lie}}(\mathfrak{g}, \mathcal{M}).$$

Proof. Since $L = \mathrm{A} \otimes \mathfrak{g}$ we have the isomorphisms

$$\Lambda^n_{\mathrm{A}} L \otimes_{\mathrm{A}} \mathcal{M} \cong \Lambda^n_{\mathrm{A}} \mathfrak{g} \otimes_{\mathrm{A}} \mathcal{M}$$
$$\mathsf{Hom}_{\mathrm{A}}(\Lambda^n_{\mathrm{A}} L, \mathcal{M}) \cong \mathsf{Hom}(\Lambda^n \mathfrak{g}, \mathcal{M})$$

and the proof is straightforward. $\square$

Proposition 3.20. *Let L be a free Lie–Rinehart algebra generated by $\psi \colon \mathcal{N} \to \mathtt{Der}_K(\mathrm{A})$ and let $\mathcal{M}$ be any Lie–Rinehart* (A, L)*-module. Then*

$$H^{\mathrm{Rin}}_n(L, \mathcal{M}) = 0, \qquad n > 1,$$
$$H^n_{\mathrm{Rin}}(L, \mathcal{M}) = 0, \qquad n > 1.$$

Proof. By construction L is a transformation Lie–Rinehart algebra of $(\mathrm{F}(\mathcal{N}), \mathrm{A})$. Thus we can apply Lemma 3.19 to get the isomorphisms $H_n^{\mathrm{Rin}}(L, \mathcal{M}) \cong H_n^{\mathrm{Lie}}(\mathrm{F}(\mathcal{N}), \mathcal{M})$ and $H^n_{\mathrm{Rin}}(L, \mathcal{M}) \cong H^n_{\mathrm{Lie}}(\mathrm{F}(\mathcal{N}), \mathcal{M})$ so the proof follows by Proposition 3.6. □

Now we will describe the low homology of Lie–Rinehart algebras. By definition,

$$H_0^{\mathrm{Rin}}(L, \mathcal{M}) = \mathcal{M}_L = \frac{\mathcal{M}}{\mathcal{M} \circ L},$$

is the module of *coinvariants* of $\mathcal{M}$, where $\mathcal{M} \circ L$ means the K-submodule of $\mathcal{M}$ generated by mx, for all $m \in \mathcal{M}$ and $x \in L$. In the same way, the *invariant K-submodule* of $\mathcal{M}$ is

$$H^0_{\mathrm{Rin}}(L, \mathcal{M}) = \mathcal{M}^L = \{m \in \mathcal{M} \mid mx = 0 \text{ for all } x \in L\}.$$

Proposition 3.21. *If $\mathcal{M}$ is any trivial right (A, L)-module, then*

$$H_1^{\mathrm{Rin}}(L, \mathcal{M}) \cong \frac{\mathcal{M} \otimes_{\mathrm{A}} L}{\mathcal{M} \otimes_{\mathrm{A}} \{L, L\}} \cong L^{\mathrm{ab}} \otimes \mathcal{M}.$$

Definition 3.22. If $\mathcal{M}$ is a left (A, L)-module, we denote by $\mathsf{Der}_{\mathrm{A}}(L, \mathcal{M})$ the A-linear maps $D\colon L \to \mathcal{M}$ which are derivations from the Lie K-algebra L to $\mathcal{M}$. In other words, the map D must satisfy

$$D(ax) = aD(x)D([x, y]) = x(Dy) - y(Dx).$$

for all $a \in \mathrm{A}$ and $x, y \in L$.

An *inner derivation* is a derivation of $\mathsf{Der}_{\mathrm{A}}(L, \mathcal{M})$ defined by $D_m(x) = xm$. The K-submodule of inner derivations is denoted by $\mathsf{IDer}_{\mathrm{A}}(L, \mathcal{M})$ and it is a K-submodule of $\mathsf{Der}_{\mathrm{A}}(L, \mathcal{M})$.

Proposition 3.23. *If $\mathcal{M}$ is a left (A, L)-module,*

$$H^1_{\mathrm{Rin}}(L, \mathcal{M}) \cong \frac{\mathsf{Der}_{\mathrm{A}}(L, \mathcal{M})}{\mathsf{IDer}_{\mathrm{A}}(L, \mathcal{M})}.$$

Corollary 3.24. *If $\mathcal{M}$ is a trivial left (A, L)-module,*

$$H^1_{\mathrm{Rin}}(L, \mathcal{M}) \cong \mathsf{Der}_{\mathrm{A}}(L, \mathcal{M}) \cong \mathsf{Hom}_{\mathrm{Rin}}(L, \mathcal{M}) \cong \mathsf{Hom}_{\mathrm{A}}(L^{\mathrm{ab}}, \mathcal{M}).$$

Definition 3.25. Let L be a Lie–Rinehart A-algebra and let $\mathcal{M}$ a left Lie-Rinehart (A, L)-module. An *abelian extension* of L by $\mathcal{M}$ is an exact sequence

$$0 \longrightarrow \mathcal{M} \xrightarrow{i} L' \xrightarrow{\partial} L \longrightarrow 0,$$

where L' is a Lie–Rinehart algebra over A and ∂ is a Lie–Rinehart algebra homomorphism. Moreover, i is an A-linear map and the following identities hold

$$\begin{aligned} [i(m), i(n)] &= 0, \\ [i(m), x'] &= \big(\partial(x')\big)(m), \qquad m, n \in \mathcal{M},\ x' \in L'. \end{aligned}$$

The classification of abelian extensions can be found in [14].

Theorem 3.26. *If L is* A*-projective, then the cohomology $H^2_{\mathrm{Rin}}(L,\mathcal{M})$ classifies the abelian extensions*

$$0 \longrightarrow \mathcal{M} \longrightarrow L' \longrightarrow L \longrightarrow 0$$

of L by $\mathcal{M}$ in the category of Lie–Rinehart algebras which split in the category of A*-modules. Moreover, the extension*

$$0 \longrightarrow \mathcal{M} \longrightarrow L \rtimes \mathcal{M} \longrightarrow L \longrightarrow 0$$

represents the $0 \in H^2_{\mathrm{Rin}}(L,\mathcal{M})$.

Chapter 4

Universal Central Extensions and Tensor Product

In this chapter we will give the construction of the Universal Central Extension of a Lie–Rinehart algebra and we will give a definition of the non-abelian tensor product to end relating these two objects.

4.1 Central Extensions

Definition 4.1. An *extension* of a Lie–Rinehart algebra L is a short exact sequence

$$0 \longrightarrow I \xrightarrow{i} E \xrightarrow{p} L \longrightarrow 0,$$

where I, E and L are Lie–Rinehart algebras and i, p are Lie–Rinehart homomorphisms. Since $i\colon I \to i(I) = \mathsf{Ker}\, p$ is an isomorphism we shall identify I and $i(I)$. In other words, an extension of L is an surjective Lie–Rinehart homomorphism $p\colon E \to L$. If $p\colon E \to L$ and $p'\colon E' \to L$ are two extensions of L, a *homomorphism* from p to p' is a commutative diagram in $\mathtt{LR}_{\mathrm{AK}}$ of the form

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & L & \end{tikzcd}$$

In particular, one have the following relations

$$\mathsf{Ker}\, f \subset f^{-1}(\mathsf{Ker}\, p') = \mathsf{Ker}\, p,$$
$$E' = f(E) + \mathsf{Ker}\, p'.$$

Definition 4.2. Given an extension, one says that *splits* if there exists a Lie–Rinehart homomorphism $s\colon L \to E$, called *splitting homomorphism*, such that $ps = 1_L$. In this case, $E = I \oplus s(L)$ and $s\colon L \to s(L)$ is an isomorphism with inverse $f|s(L)$. Moreover, $K \simeq I \rtimes L$, the semidirect product. In this way, semidirect products and split exact sequences are in a one-to-one correspondence. We point out that not every extension splits. We shall say that an extension *splits uniquely* whenever the splitting morphism is unique.

Definition 4.3. A *central extension of* L is an extension such that $\mathsf{Ker}\, p \subset \mathsf{Z}_{\mathrm{A}}(E)$. In particular, if $p\colon\ E \underset{p}{\overset{s}{\rightleftarrows}} L$ is a split central extension is a direct product of K-Lie algebras $E = \mathsf{Ker}\, p \times L$, which is also a Lie–Rinehart algebra.

Proposition 4.4. *If* L *is* A*-projective, then* $H^2_{\mathrm{Rin}}(L, I)$ *classifies the central extensions*

$$0 \longrightarrow I \longrightarrow E \longrightarrow L \longrightarrow 0$$

of L *by* I*.*

Proof. Note that, if I is a trivial left Lie–Rinehart (A, L)-module, then an abelian extension of L by I is a central extension, and so the assertion follows by Proposition 3.26. $\square$

Definition 4.5. A Lie–Rinehart algebra over A is said *perfect* if $L = \{L, L\}$. A central extension E of L is called a *covering* if E is perfect and in this case, L is also perfect.

Definition 4.6. A central extension $\mathfrak{u}\colon \mathcal{L} \twoheadrightarrow L$ is called *universal* if there exists a unique homomorphism from $\mathfrak{u}$ to any other central extension of L. From this property it immediately follows that two universal central extensions of L are isomorphic as extensions.

Lemma 4.7. ***(central trick)*** *Let* $p\colon E \twoheadrightarrow L$ *be a central extension.*

(a) If $p(x) = p(x')$ *and* $p(y) = p(y')$ *then* $[x, y] = [x', y']$ *and for every* $a \in \mathrm{A}$*,* $x(a) = x'(a)$*.*

(b) If the following diagram commutes in $\mathtt{LR}_{\mathrm{AK}}$*,*

$$P \underset{g}{\overset{f}{\rightrightarrows}} E \overset{p}{\twoheadrightarrow} L$$

then the restriction of both f *and* g *to* $\{P, P\}$ *agree; i.e.,* $f|_{\{P,P\}} = g|_{\{P,P\}}$*.*

Proof. (a) Since $p(x) = p(x')$, we have that $x' = x + z$ for some $z \in \mathsf{Ker}\, p \subset \mathsf{Z}_{\mathrm{A}}(E)$, and the same for $y' = y + z'$. In this way, $[x', y'] = [x + z, y + z'] = [x, y]$ since $z, z' \in \mathsf{Z}_{\mathrm{A}}(E)$. In addition, since p is a Lie–Rinehart homomorphism, the action on $\mathtt{Der}_K(\mathrm{A})$ must be preserved so $x(a) = x(a')$ for all $a \in \mathrm{A}$.

(b) Since $p\big(f(x)\big) = p\big(g(x)\big)$, using part (a), we have

$$f(a[x, y]) = a[f(x), f(y)] = a[g(x), g(y)] = g(a[x, y]).$$

□

Lemma 4.8. *Let L be perfect and $p\colon E \twoheadrightarrow L$ be a central extension.*

(a) $E = \{E,E\} + \mathsf{Ker}\, p$, and $p' = p_{\{E,E\}}\colon \{E,E\} \twoheadrightarrow L$ is a covering.

(b) $\mathsf{Z}_{\mathrm{A}}\, E = p^{-1}(\mathsf{Z}_{\mathrm{A}}\, L)$ and $p(\mathsf{Z}_{\mathrm{A}}\, E) = \mathsf{Z}_{\mathrm{A}}\, L$.

(c) If $f\colon L \twoheadrightarrow M$ is a central extension then so is $fp\colon E \twoheadrightarrow M$.

(d) If $f\colon C \twoheadrightarrow L$ is a covering and

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, two heads, "p"'] & & C \arrow[dl, two heads, "f"] \\ & L & \end{tikzcd}$$

a morphism of extensions, then $g\colon E \twoheadrightarrow C$ is a central extension. In particular, g is surjective.

Proof. (a) Since $p(\{E,E\}) = \{L,L\} = L$, it follows that $E = \{E,E\} + \mathsf{Ker}\, p$. Moreover, $p_{\{E,E\}}$ is surjective so it is a covering.

(b) Let $z \in \mathsf{Z}_{\mathrm{A}}(E)$. For every $a \in \mathrm{A}$ we have that $[az, E] = 0$, so $0 = [p(az), p(E)] = [ap(z), L]$ then $p(z) \in \mathsf{Z}_{\mathrm{A}}(L)$ and in conclusion $z \in p^{-1}\big(\mathsf{Z}_{\mathrm{A}}(L)\big)$. Conversely, let $z \in p^{-1}\big(\mathsf{Z}_{\mathrm{A}}(L)\big)$, for every $a \in \mathrm{A}$ we have that $p([az, E]) = [ap(z), L] = 0$ so $[az, E] \subset \mathsf{Ker}\, p \subset \mathsf{Z}_{\mathrm{A}}(E)$. Since $[az, E] = [az, \{E,E\} + \mathsf{Ker}\, p] = [az, \{E,E\}]$ we just have to check that $[az, \{E,E\}]$ is zero. Therefore, by the Jacobi identity and since $z(b) = 0$ for all $b \in \mathrm{A}$, $\big[az, b[x,y]\big] = b\big[az, [x,y]\big] = b\big[x, [az,y]\big] + b\big[y, [x, az]\big] = 0$.

(c) The morphism fp is composition of surjective maps, so it is surjective. Moreover, $\mathsf{Ker}\, fp = p^{-1}(\mathsf{Ker}\, f) \subset p^{-1}\big(\mathsf{Z}_{\mathrm{A}}(L)\big) = \mathsf{Z}_{\mathrm{A}}(E)$.

(d) By Lemma 4.7(b) we have that $C = \{C,C\} = \{g(E) + \mathsf{Ker}\, f, g(E) + \mathsf{Ker}\, f\} = \{g(E), g(E)\} = g(\{E,E\})$, so it is surjective. In addition it is central since $\mathsf{Ker}\, g \subset \mathsf{Ker}\, p$. □

Corollary 4.9. *Let L a Lie–Rinehart algebra. If $L/\,\mathsf{Z}_{\mathrm{A}}\, L$ is perfect, then $\mathsf{Z}_{\mathrm{A}}(L/\,\mathsf{Z}_{\mathrm{A}}\, L) = 0$.*

Proof. We are in conditions to apply the second formula of Lemma 4.8(b) to the canonical map $p\colon L \twoheadrightarrow L/\,\mathsf{Z}_{\mathrm{A}}(L)$. Then, $\mathsf{Z}_{\mathrm{A}}\big(L/\,\mathsf{Z}_{\mathrm{A}}(L)\big) = p\big(\mathsf{Z}_{\mathrm{A}}(L)\big) = 0$. □

In particular for a perfect L this corollary says that $L/\,\mathsf{Z}_{\mathrm{A}}(L)$ is the "smallest" central quotient.

Lemma 4.10. ***(Pullback Lemma)*** *Let $c\colon N \twoheadrightarrow M$ be a central extension and $f\colon L \to M$ a morphism of Lie–Rinehart algebras, then,*

$$P := \{(x,n) \in L \times_{\mathtt{Der}_K(\mathrm{A})} N : f(x) = c(n)\}$$

is a Lie–Rinehart algebra and $p_L\colon P \to L$, $(x,n) \mapsto x$, *is a central extension. This extension splits if and only if there exists a (unique) Lie–Rinehart morphism* $h\colon L \to N$ *such that* $ch = f$.

$$\begin{tikzcd} P \arrow[r, "p_N"] \arrow[d, two heads, "p_L"'] & N \arrow[d, two heads, "c"] \\ L \arrow[r, "f"'] \arrow[u, dotted, bend left, "s"] \arrow[ur, dotted, "h"] & M \end{tikzcd}$$

Proof. The pullback P inherit the canonical Lie–Rinehart algebra structure, and p_L is a central extension since it is clearly surjective and if $(x,n) \in \operatorname{Ker} p_L$ then $x = 0$ and $c(n) = 0$, so $[a(0,n),(y,m)] = ([0,y],[an,m]) = 0$ for all $y \in L$ and $m \in N$.

If p_L splits we define $h = p_N \circ s$. Conversely, given $h\colon L \to N$ we define $s\colon L \to P$ such that $x \mapsto \big(x, h(x)\big)$. We will see now that s is a Lie–Rinehart algebra morphism checking that it is a Lie morphism, an A-modules morphism and the anchor map is conserved:

- $s([x,y]) = \big([x,y], h([x,y])\big) = \big[\big(x,h(x)\big),\big(y,h(y)\big)\big] = [s(x),s(y)]$,
- $s(ax) = \big(ax, h(ax)\big) = a\big(x,h(x)\big) = as(x)$,
- $s(x)(a) = \big(x,h(x)\big)(a) = h(x)(a)$,

for all $a \in \mathrm{A}$ and $x, y \in L$. □

Theorem 4.11. ***(characterization of universal central extensions)*** *Given a Lie–Rinehart algebra* L*, there are equivalent:*

(1) Every central extension $L' \twoheadrightarrow L$ *splits uniquely.*

(2) $1_L\colon L \twoheadrightarrow L$ *is a universal central extension.*
If $\mathfrak{u}\colon L \twoheadrightarrow M$ *is a central extension, then* (1) *and* (2) *are equivalent to*

(3) $\mathfrak{u}\colon L \twoheadrightarrow M$ *is a universal central extension of* M*. In this case,*

(a) both L *and* M *are perfect and*

(b) $\mathsf{Z}_{\mathrm{A}} L = \mathfrak{u}^{-1}(\mathsf{Z}_{\mathrm{A}} L) = \mathsf{Z}_{\mathrm{A}} M$.

Proof. Clearly 1_L is a central extension so to see that (1) is equivalent to (2), it is enough to see that every central $p\colon L' \twoheadrightarrow L$ splits uniquely if and only if there exists a unique homomorphism $f\colon L \twoheadrightarrow L'$ such that $pf = 1_L$, which is the definition of 1_L being a universal central extension.

Suppose now that (3) holds. Let $L \times L/\{L,L\}$ be the product as Lie algebras over K. It inherits also the A-module structure, since $\{L,L\}$ is also an A-module. The only problem is the equality relating the A-module structure and the Lie algebra, but we will see that in this case it is also conserved, so it is a Lie–Rinehart algebra over A.

$$[(x, y + \{L, L\}), a(x', y' + \{L, L\})] = (a[x, x'], 0) + (x(a)x', 0),$$

and

$$a[(x, y + \{L, L\}), (x', y' + \{L, L\})] = (a[x, x'], 0),$$

$$\begin{aligned}(x, y + \{L, L\})(a)(x', y' + \{L, L\}) &= (x(a)x', x(a)y' + \{L, L\})\\ &= (x(a)x', [x, ay'] - a[x, y'] + \{L, L\})\\ &= (x(a)x', 0).\end{aligned}$$

Now we can define the central extension $\bar{\mathfrak{u}}\colon L \times L/\{L, L\} \twoheadrightarrow M$, and two maps f and g

$$\begin{tikzcd} L \arrow[rr, "\mathfrak{u}"] \arrow[dr, shift left, "f"] \arrow[dr, shift right, "g"'] & & M \\ & L \times L/\{L, L\} \arrow[ur, "\bar{\mathfrak{u}}"'] & \end{tikzcd}$$

where $f(x) = (x, x + \{L, L\})$ and $g(x) = (x, 0)$. Since $\mathfrak{u}$ is universal, f and g must be equal, so $L/\{L, L\} = 0$ so L is perfect. By the surjectivity of $\mathfrak{u}$, M is perfect too. The assertion (b) is an immediate consequence of Lemma 4.8(b).

We can prove now that (3) ⇒ (1), since given a central extension $f\colon L' \twoheadrightarrow M$ we can apply Lemma 4.8(c) so $\mathfrak{u}f$ is a central extension too. By the universality of $\mathfrak{u}$, there exists $f\colon L \twoheadrightarrow L'$ such that $\mathfrak{u}fg = \mathfrak{u}$ and by Lemma 4.7(b) we have that $fg = 1_L$.

To see that (1) ⇒ (3), given a central extension $f\colon N \twoheadrightarrow M$ we construct as in Lemma 4.10 the central extension $p_L\colon P \twoheadrightarrow L$, which by assumption splits uniquely. Therefore by Lemma 4.10 there exists a unique Lie–Rinehart morphism $h\colon L \twoheadrightarrow N$ so $\mathfrak{u}\colon L \twoheadrightarrow M$ is a universal central extension. □

Corollary 4.12. *Let $f \twoheadrightarrow E \to L$ and $g \twoheadrightarrow L \to M$ be central extensions. Then $gf\colon E \to M$ is a universal central extension if and only if f is a universal central extension.*

Proof. Since E is perfect, we can apply Lemma 4.8(c) so the extension gf is central. Hence, f is universal if and only if $1_E\colon E \twoheadrightarrow E$ is universal, if and only if gf is universal. □

Corollary 4.13. *Let L and L' be perfect Lie–Rinehart algebras, with universal central extensions $\mathfrak{u}\colon \mathcal{L} \twoheadrightarrow L$ and $\mathfrak{u}'\colon \mathcal{L}' \twoheadrightarrow L'$ respectively. Then*

$$L/\,\mathsf{Z}_{\mathrm{A}}(L) \cong L'/\,\mathsf{Z}_{\mathrm{A}}(L') \iff \mathcal{L} \cong \mathcal{L}'.$$

Proof. Given the diagram

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "\mathfrak{u}"] \arrow[d, "\phi"'] & L \arrow[r, "\pi"] & L/\,\mathsf{Z}_{\mathrm{A}}(L) \arrow[d, "\varphi"] \\
\mathcal{L}' \arrow[r, "\mathfrak{u}'"] & L' \arrow[r, "\pi'"] & L/\,\mathsf{Z}_{\mathrm{A}}(L'),
\end{tikzcd}$$

we know that ϕ exists and is an isomorphism if and only if φ exists and is an isomorphism. By Corollary 4.12, the maps $\pi\mathfrak{u}$ and $\pi'\mathfrak{u}'$ are universal central extensions and since $L/\,\mathsf{Z}_{\mathrm{A}}(L)$ is isomorphic to $L'/\,\mathsf{Z}_{\mathrm{A}}(L')$, by the uniqueness of the universal central extension, $\mathcal{L} \cong \mathcal{L}'$. Conversely, by Corollary 4.9 $L/\,\mathsf{Z}_{\mathrm{A}}(L)$ is centreless. By Lemma 4.8 (b) $\mathsf{Z}_{\mathrm{A}}(\mathcal{L}) = \mathsf{Ker}(\pi\mathfrak{u})$ and $\mathsf{Z}_{\mathrm{A}}(\mathcal{L}') = \mathsf{Ker}(\pi'\mathfrak{u}')$. Therefore, $\mathsf{Ker}(\pi'\mathfrak{u}'\phi) = \phi^{-1}\big(\mathsf{Ker}(\pi'\mathfrak{u}')\big) = \phi^{-1}\big(\mathsf{Z}_{\mathrm{A}}(\mathcal{L}')\big) = \mathsf{Z}_{\mathrm{A}}(\mathcal{L}) = \mathsf{Ker}(\pi\mathfrak{u})$. Since $\pi\mathfrak{u}$ and $\pi'\mathfrak{u}'\phi$ are surjective, φ exists and is an isomorphism. □

4.2 The Universal Central Extension

In this section, we will prove that if L is a perfect Lie–Rinehart algebra over A it has a universal central extension, and we will give it explicitly.

Let L be a Lie–Rinehart algebra over A. We denote by $M_{\mathrm{A}}L$ the A-submodule of $\mathrm{A} \otimes_K L \otimes_K L$ spanned by the elements of the form

1. $a \otimes x \otimes x$
2. $a \otimes x \otimes y + a \otimes y \otimes x$
3. $a \otimes x \otimes [y, z] + a \otimes y \otimes [z, x] + a \otimes z \otimes [x, y]$
4. $a \otimes [x, y] \otimes [x', y'] + [x, y](a) \otimes x' \otimes y' - 1 \otimes [x, y] \otimes a[x', y']$

with $x, x', y, y', z \in L$ and $a \in \mathrm{A}$, and we define

$$\mathfrak{uce}_{\mathrm{A}}L := \mathrm{A} \otimes_K L \otimes_K L/M_{\mathrm{A}}L.$$

writing $(a, x, y) := a \otimes x \otimes y + M_{\mathrm{A}}L \in \mathfrak{uce}_{\mathrm{A}}L$.

We recall that by construction, the following identities hold in $\mathfrak{uce}_{\mathrm{A}}$:

1. $(a, x, y) = -(a, y, x)$,
2. $(a, x, [y, z]) + (a, y, [z, x]) + (a, z, [x, y]) = 0$,
3. $(1, [x, y], a[x', y']) = (a, [x, y], [x', y']) + ([x, y](a), x', y')$.

The map of A-modules $\mathrm{A} \otimes_K L \otimes_K L \to L$, determined by $(a, x, y) \mapsto a[x, y]$, vanishes on $M_{\mathrm{A}} L$ so it descends to a linear map

$$\mathfrak{u} \colon \mathfrak{uce}_{\mathrm{A}} L \to L.$$

It is a tedious but straightforward calculation to check that $\mathfrak{uce}_{\mathrm{A}} L$ is a Lie algebra with product

$$[(a, x, y), (a', x', y')] := (aa', [x, y], [x', y']) + (a[x, y](a'), x', y') - ([x', y'](a)a', x, y),$$

In addition, it is clearly an A-module, so defining the anchor map as

$$(a, x, y)(b) := a[x, y](b),$$

we check that $\mathfrak{uce}_{\mathrm{A}} L$ is a Lie–Rinehart algebra watching if it follows the identity relating both structures:

$$\begin{aligned}
&[(a, x, y), b(a', x', y')] = (aa'b, [x, y], [x', y']) + (a[x, y](a'b), x', y') - ([x', y'](a)a'b, x, y) \\
&= b(aa', [x, y], [x', y']) + b(a[x, y](a'), x', y') - b([x', y'](a)a', x, y) + (aa'[x, y](b), x', y') \\
&= b[(a, x, y), (a', x', y')] + a[x, y](b)(a', x', y').
\end{aligned}$$

Moreover, it is easy to check that the map $\mathfrak{u} : \mathfrak{uce}_{\mathrm{A}} L \twoheadrightarrow \{L, L\}$ is a central extension of $\{L, L\}$. Now let $f \colon L \to M$ a Lie–Rinehart algebra homomorphism. Let $M_{\mathrm{A}} M \in \mathrm{A} \otimes_K M \otimes_K M$ defined analogously to $M_{\mathrm{A}} L$. The map $1_{\mathrm{A}} \otimes_K f \otimes_K f \colon M_{\mathrm{A}} L \to M_{\mathrm{A}} M$ induces an A-linear map

$$\mathfrak{uce}_{\mathrm{A}}(f) \colon \mathfrak{uce}_{\mathrm{A}} L \to \mathfrak{uce}_{\mathrm{A}} M, \quad (a, x, y) \mapsto \big(a, f(x), f(y)\big).$$

We want to check that $\mathfrak{uce}_{\mathrm{A}}(f)$ is a Lie–Rinehart algebra morphism, but since the anchor map is preserved by f, we have that

$$a[x, y](a') = f(a[x, y])(a') = a[f(x), f(y)](a'),$$

so it follows that

$$\begin{aligned}
&\mathfrak{uce}_{\mathrm{A}}(f)([(a, x, y), (a', x', y')]) \\
&\qquad = \big(aa', f([x, y]), f([x', y'])\big) + \big(a[x, y](a'), f(x'), f(y')\big) - \big([x', y'](a)a', f(x), f(y)\big) \\
&\qquad = [\mathfrak{uce}_{\mathrm{A}}(f)(a, x, y), \mathfrak{uce}_{\mathrm{A}}(f)(a', x', y')].
\end{aligned}$$

Moreover, the following diagram commutes by construction

$$\begin{tikzcd}
\mathfrak{uce}_{\mathrm{A}} L \arrow[r, "\mathfrak{uce}_{\mathrm{A}}(f)"] \arrow[d, "\mathfrak{u}_L"'] & \mathfrak{uce}_{\mathrm{A}} M \arrow[d, "\mathfrak{u}_M"] \\
L \arrow[r, "f"'] & M.
\end{tikzcd}$$

Proposition 4.14. *Let $f\colon L \to M$ be a morphism of Lie–Rinehart algebras and suppose that $g\colon M' \to M$ is a central extension. Then there exists a homomorphism $\mathfrak{f}\colon \mathfrak{uce}_{\mathrm{A}} L \to M'$, making the following diagram commutative*

$$\begin{tikzcd} \mathfrak{uce}_{\mathrm{A}} L \arrow[r, "\mathfrak{f}"] \arrow[d, "\mathfrak{u}"'] & M' \arrow[d, "g"] \\ L \arrow[r, "f"'] & M. \end{tikzcd}$$

The map $\mathfrak{f}$ is uniquely determined on $\{\mathfrak{uce}_{\mathrm{A}} L, \mathfrak{uce}_{\mathrm{A}} L\}$ by the commutativity of the diagram.

Proof. Let $s\colon M \to M'$ be a section of g in $\mathtt{Set}$. The map s may not be linear but we know that $s(km) - ks(m) \in \mathsf{Ker}\, g \subset \mathsf{Z}_{\mathrm{A}}(M')$ and $s(m+n) - s(m) - s(n) \in \mathsf{Ker}\, g \subset \mathsf{Z}_{\mathrm{A}}(M')$ for $k \in K$ and $m, n \in M$. Using this, we can say that the map

$$\begin{aligned} \mathrm{A} \times L \times L &\xrightarrow{\bar{f}} M' \\ (a, x, y) &\longmapsto a[sf(x), sf(y)], \end{aligned}$$

is bilinear, since

$$a[sf(kx), sf(y)] = a[sf(kx) - ksf(x) + ksf(x), sf(y)] = a[ksf(x), sf(y)],$$

$$\begin{aligned} a[s\big(f(x+y)\big), sf(z)] &= a[s\big(f(x) + f(y)\big) - s\big(f(x)\big) - s\big(f(y)\big) + s\big(f(x)\big), sf(z)] \\ &= a[sf(x), sf(z)] + a[sf(y), sf(z)]. \end{aligned}$$

By the universal property of tensor product, $\bar{f}$ defines a unique map between $\mathrm{A} \otimes_K L \otimes_K L$ and M'. In addition, the map is zero in $M_{\mathrm{A}} L$, so it can be extended to $\mathfrak{f}\colon \mathfrak{uce}_{\mathrm{A}} L \to M'$, making the diagram commutative. This map conserves the anchor map because the section s must conserve it too. Using the property that $a[x, y](a') = f(a[x, y])(a') = a[f(x), f(y)](a')$, it follows immediately that $\mathfrak{f}$ is a Lie algebra homomorphism hence it is a Lie–Rinehart algebra homomorphism, that makes the diagram commutative. The uniqueness in $\{\mathfrak{uce}_{\mathrm{A}} L, \mathfrak{uce}_{\mathrm{A}} L\}$ follows from Lemma 4.7(b). □

Theorem 4.15. *Let L be a perfect Lie–Rinehart algebra. Supposing that A has a right (A, L)-module structure, then*

$$0 \longrightarrow H_2^{\mathrm{Rin}}(L, A) \longrightarrow \mathfrak{uce}_{\mathrm{A}} L \xrightarrow{\mathfrak{u}} L \longrightarrow 0$$

is a universal central extension of L.

Proof. It can be seen that $\mathfrak{uce}_{\mathrm{A}}(\{L, L\}) \subset \{\mathfrak{uce}_{\mathrm{A}} L, \mathfrak{uce}_{\mathrm{A}} L\} \subset \mathfrak{uce}_{\mathrm{A}} L$. Thus when L is perfect, $\{\mathfrak{uce}_{\mathrm{A}} L, \mathfrak{uce}_{\mathrm{A}} L\} = \mathfrak{uce}_{\mathrm{A}} L$, so applying Proposition 4.14 for every central extension $f\colon M \twoheadrightarrow L$ we have a unique map $\mathfrak{f}\colon \mathfrak{uce}_{\mathrm{A}} L \to M$ making the diagram commutative. In other words, $\mathfrak{uce}_{\mathrm{A}} L$ is the universal central extension of L. □

4.3 Non-abelian Tensor Product

In Lie algebras, the non-abelian tensor product was introduced by Ellis in [10]. We will give first the definition in Lie algebras and then we will generalize it to Lie–Rinehart algebras, finding some properties and relating it to the universal central extension.

Definition 4.16. Let L, M be K-Lie algebras. By an *action* of L on M, we mean a K-linear map, $L \times M \to M$, $(x, m) \mapsto {}^{x}m$, satisfying

$$^{[x,y]}m = {}^{x}({}^{y}m) - {}^{y}({}^{x}m), \qquad {}^{x}[m,n] = [{}^{x}m, n] + [m, {}^{x}n],$$

for all $x, y \in L$ and $m, n \in M$.

For example, if L is a subalgebra of some Lie algebra $\mathcal{L}$ and M is an ideal of $\mathcal{L}$ then the bracket in $\mathcal{L}$ yields an action of L on M.

Definition 4.17. If we have an action of L on M and an action of M on L, for any Lie–Rinehart algebra $\mathcal{L}$ we call a K-bilinear function $f \colon L \times M \to \mathcal{L}$ a *Lie pairing* if

1. $f([x,y], m) = f(x, {}^{y}m) - f(y, {}^{x}m)$,
2. $f(x, [m,n]) = f({}^{n}x, m) - f({}^{m}x, n)$,
3. $f\big({}^{m}x), ({}^{y}n)\big) = -[f(x,m), f(y,n)]$,

for all $x, y \in L$ and $m, n \in M$.

We say that a Lie pairing $f \colon L \times M \to \mathcal{L}$ is *universal* if for any other Lie pairing $g \colon L \times M \to \mathcal{L}'$ there is a unique Lie homomorphism $\varphi \colon \mathcal{L} \to \mathcal{L}'$ making commutative the diagram:

$$\begin{tikzcd} L \times M \arrow[r, "f"] \arrow[dr, "g"'] & \mathcal{L} \arrow[d, "\varphi"] \\ & \mathcal{L}'. \end{tikzcd}$$

The Lie algebra $\mathcal{L}$ is unique up to isomorphism which we will describe as the non-abelian tensor product of L and M.

Definition 4.18. Let L and M be a pair of Lie algebras together with an action of L on M and an action of M on L. We define the *non-abelian tensor product of L and M*, denoted by $L \otimes M$, as the Lie algebra spanned as an K-module by the symbols $x \otimes m$, and subject only to the relations:

1. $k(x \otimes m) = kx \otimes m = x \otimes km$,

2. $x \otimes (m+n) = x \otimes m + x \otimes n$,
   $(x+y) \otimes m = x \otimes m + y \otimes m$,

3. $[x,y] \otimes m = x \otimes {}^{y}m - y \otimes {}^{x}m$,
   $x \otimes [m,n] = {}^{n}x \otimes m - {}^{m}x \otimes n$,

4. $[(x \otimes m),(y \otimes n)] = -({}^{m}x \otimes {}^{y}n))$,

for every $k \in K$, $x, y \in L$ and $m, n \in M$.

Theorem 4.19. *Given a perfect Lie algebra L, the tensor product $L \otimes L$ where the action of L on L is the Lie bracket, is the universal central extension of L, and hence, $H_2(L,K)$ is isomorphic to the kernel of the map $L \otimes L \to L$.*

Now we generalize this results to Lie–Rinehart algebras.

Definition 4.20. Let L, M be Lie–Rinehart algebras. By an *action* of L on M, we mean an K-linear map, $L \times M \to M$, $(x,m) \mapsto \ {}^{x}m$, satisfying

1. ${}^{x}(am) = a({}^{x}m) + x(a)m$,

2. ${}^{[x,y]}m = \ {}^{x}({}^{y}m) - \ {}^{y}({}^{x}m)$,

3. ${}^{x}[m,n] = [{}^{x}m,n] + [m,{}^{x}n]$,

for all $a \in \mathrm{A}$, $x, y \in L$ and $m, n \in M$.

For example, if L is a subalgebra of some Lie–Rinehart algebra $\mathcal{L}$ and M is an ideal of $\mathcal{L}$ then the bracket in $\mathcal{L}$ yields an action of L on M.

Definition 4.21. If we have an action of L on M and an action of M on L, for any Lie–Rinehart algebra $\mathcal{L}$ we call a K-bilinear function $f\colon L \times M \to \mathcal{L}$ a *Lie–Rinehart pairing* if

1. $\alpha_{\mathcal{L}}\big(f(x,m)\big) = [\alpha_L(x), \alpha_M(m)]$,

2. $f([x,y],m) = f(x,{}^{y}m) - f(y,{}^{x}m)$,

3. $f(x,[m,n]) = f({}^{n}x,m) - f({}^{m}x,n)$,

4. $$f\big(a({}^{m}x), b({}^{y}n)\big) = -ab[f(x,m),f(y,n)] - a[\alpha_L(x),\alpha_M(m)](b)f(y,n) + [\alpha_L(y),\alpha_M(n)](a)bf(x,m),$$

for all $a, b \in \mathrm{A}$, $x, y \in L$ and $m, n \in M$.

Definition 4.22. We say that a Lie–Rinehart pairing $f\colon L \times M \to \mathcal{L}$ is *universal* if for any other Lie–Rinehart pairing $g\colon L \times M \to \mathcal{L}'$ there is a unique Lie–Rinehart homomorphism $\varphi\colon \mathcal{L} \to \mathcal{L}'$ making commutative the diagram:

$$\begin{tikzcd} L \times M \arrow[r, "f"] \arrow[dr, "g"'] & \mathcal{L} \arrow[d, "\varphi"] \\ & \mathcal{L}'. \end{tikzcd}$$

The Lie–Rinehart algebra $\mathcal{L}$ is unique up to isomorphism which we will describe as the non-abelian tensor product of L and M.

Definition 4.23. Let L and M be a pair of Lie–Rinehart algebras together with an action of L on M and an action of M on L. We define the *non-abelian tensor product of L and M in* $\mathtt{LR}_{\mathrm{AK}}$, $L \otimes M$, as the Lie–Rinehart algebra over A spanned as an A-module by the symbols $x \otimes m$, and subject only to the relations:

1. $k(x \otimes m) = kx \otimes m = x \otimes km$,

2. $x \otimes (m + n) = x \otimes m + x \otimes n$,
   $(x + y) \otimes m = x \otimes m + y \otimes m$,

3. $[x, y] \otimes m = x \otimes {}^{y}m - y \otimes {}^{x}m$,
   $x \otimes [m, n] = {}^{n}x \otimes m - {}^{m}x \otimes n$,

4. $[a(x \otimes m), b(y \otimes n)] = -ab({}^{m}x \otimes {}^{y}n) + a\alpha(x \otimes m)(b)(y \otimes n) - \alpha(y \otimes n)(a)b(x \otimes m)$,

for every $k \in K$, $a, b \in \mathrm{A}$, $x, y \in L$ and $m, n \in M$. Here the map $\alpha\colon L \otimes M \to \mathtt{Der}_K(\mathrm{A})$ is given by $\alpha\big(a(x \otimes m)\big) := a[\alpha_L(x), \alpha_M(m)]$.

This way, the map $f\colon L \times M \to L \otimes M$ which sends (x, m) to $x \otimes m$ is a universal Lie–Rinehart pairing by construction.

Definition 4.24. Two actions $L \times M \to M$ and $M \times L \to L$ are said to be *compatible* if for all $x, y \in L$ and $m, n \in M$,

1. $-\alpha_L({}^{m}x) = \alpha_M({}^{x}m) = [\alpha_L(x), \alpha_M(m)]$,

2. ${}^{({}^{m}x)}n = [n, {}^{x}m]$,

3. ${}^{({}^{x}m)}y = [y, {}^{m}x]$,

for all $x, y \in L$ and $m, n \in M$.

This is the case, for example, if L and M are both ideals of some Lie–Rinehart algebra and the actions are given by multiplication. We can see another example of compatible actions when $\partial\colon L \to N$ and $\partial'\colon M \to N$ are crossed modules. In this case, L and M act on each other via the action of N. These actions are compatible.

From this point on we shall assume that all actions are compatible.

Proposition 4.25. *Let $\mu\colon L \otimes M \to L$ and $\nu\colon L \otimes M \to M$ the homomorphisms defined on generators by $\mu\big(a(x \otimes m)\big) = -a({}^m x)$ and $\nu\big(a(x \otimes m)\big) = a({}^x m)$ are Lie–Rinehart homomorphisms and the following diagram is commutative:*

$$\begin{tikzcd} L \otimes M \arrow[r, "\nu"] \arrow[d, "\mu"'] \arrow[dr, "\alpha"] & M \arrow[d, "\alpha_M"] \\ L \arrow[r, "\alpha_L"'] & \mathtt{Der}_K(\mathrm{A}). \end{tikzcd}$$

We can relate the Lie–Rinehart tensor product $L \otimes M$ with the tensor product of L and M as an A-module. We will denote it by $L \underset{\mathrm{mod}}{\otimes} M$ the K-module and A-module generated by the symbols $x \otimes m$ subject to the relations

1. $k(x \otimes m) = kx \otimes m = x \otimes km$,

2. $x \otimes (m + n) = x \otimes m + x \otimes n$,
   $(x + y) \otimes m = x \otimes m + y \otimes m$,

for every $k \in K$, $x, y \in L$ and $m, n \in M$.

Proposition 4.26. *The canonical map $L \underset{\mathrm{mod}}{\otimes} M \to L \otimes M$ is a* A*-module homomorphism and is surjective. In addition, if L and M act trivially on each other, there is an isomorphism of* A*-modules:*

$$L \otimes M \cong L^{\mathrm{ab}} \underset{\mathrm{mod}}{\otimes} M^{\mathrm{ab}}.$$

Proof. If L acts trivially on M we have that $x(a)m = 0$ for $a \in \mathrm{A}, x \in L$ and $m \in M$. This means that

$$a[x, y] \otimes m = x \otimes a({}^y m) + x \otimes y(a)m - ay \otimes {}^x m - x(a)y \otimes m = 0$$

being straightforward the isomorphism. □

Proposition 4.27. *The Lie–Rinehart algebras $L \otimes M$ and $M \otimes L$ are isomorphic.*

Proof. The map $f\colon L \times M \to M \otimes L$ which sends $(x, m) \to m \otimes x$ is a Lie–Rinehart pairing, then by the universal property of $L \otimes M$ there is a Lie–Rinehart homomorphism $L \otimes M \to M \otimes L$. In a similar way, we can construct the inverse $M \otimes L \to L \otimes M$ and establish an isomorphism. □

Proposition 4.28. *Given a short exact sequence of Lie–Rinehart algebras*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

and let P be a Lie–Rinehart algebra which acts compatibly on L, M and N, and the Lie–Rinehart algebras L, M, N also act compatibly on P. In addition, the Lie–Rinehart morphisms f, g conserve these actions, i.e., $f({}^p m) = {}^p f(m)$ and ${}^m p = {}^{f(m)} p$. In this situation, the following sequence is exact

$$L \otimes P \xrightarrow{f \otimes 1} M \otimes P \xrightarrow{g \otimes 1} N \otimes P \longrightarrow 0.$$

Proof. Since f and g conserve the actions is easy to see that $f \otimes 1$ and $g \otimes 1$ is a Lie–Rinehart algebra morphism. Furthermore, the morphism $g \otimes 1$ is clearly surjective, and $\mathsf{Im}(f \otimes 1) \subset \mathsf{Ker}(g \otimes 1)$. Since $fg = 0$, we have that $f(x)(a) = 0$ for every $a \in \mathrm{A}$ and $x \in L$. This means that $(f \otimes 1)(x \otimes p)(a) = [\alpha_M\big(f(x)\big), \alpha_P(p)](a) = 0$. Moreover, $\mathsf{Im}(f \otimes 1)$ is an A-module and conserves the Lie bracket since f and g conserve the actions, so $\mathsf{Im}(f \otimes 1)$ is an ideal. Then to prove the other inclusion, we will show that $M \otimes P / \mathsf{Im}(f \otimes 1) \cong N \otimes P$. Since $\mathsf{Im}(f \otimes 1) \subset \mathsf{Ker}(g \otimes 1)$ we have a natural epimorphism $\phi \colon M \otimes P / \mathsf{Im}(f \otimes 1) \to N \otimes P$. Now we define the map $\varphi \colon N \times P \to M \otimes P / \mathsf{Im}(f \otimes 1)$ such that $\varphi(n, p) = m \otimes p + \mathsf{Im}(f \otimes 1)$ where m is such that $f(m) = n$. It is easy to prove that is a Lie pairing, so by the universality of the tensor product, there exists a unique Lie–Rinehart morphism $\bar{\varphi} \colon N \otimes P \to M \otimes P / \mathsf{Im}(f \otimes 1)$, and it is straightforward that ϕ and φ are inverse morphisms. $\square$

Theorem 4.29. *Given a perfect Lie–Rinehart algebra L, the tensor product $L \otimes L$ where the action of L on L is the Lie bracket, with the additional relation*

$$(a[x, y] \otimes b[x', y']) = ab([x, y] \otimes [x', y']) - b[x', y'](a)(x \otimes y) + a[x, y](b)(x' \otimes y'),$$

where $a, b \in \mathrm{A}$ and $x, x', y, y' \in L$, denoted by $L \hat{\otimes} L$ is the universal central extension of L.

Proof. It is routine to check that $L \hat{\otimes} L \twoheadrightarrow L$ is a central extension. To see the universality, given a central extension $p : M \twoheadrightarrow L$, we pick a section in $\mathtt{Set}$ $s \colon L \to M$. We define now a map $f \colon L \times L \to M$ by $f(x, y) = [s(x), s(y)]$. Doing the same trick as in Proposition 4.14, we see that is a Lie–Rinehart pairing, so it can be extended to $L \otimes L \to M$. It is easy to see that the map vanishes in the elements of the additional relation. Since L is perfect, we saw in Lemma 4.7(b) that this map is unique. $\square$

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