Cauchy Convergence for Normed Categories

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Talk based on joint work with M.M. Clementino (University of Coimbra) and Dirk Hofmann (University of Aveiro)

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What to expect

- Normed vector spaces with all linear operators?
- Quantales as norm recipients
- Normed sets
- Normed categories
- Normed sequential convergence
- Cauchy cocompleteness
- Some key examples
- Presheaf categories
- Cauchy cocompletion
- Banach's Fixed Point Theorem
- To-Do list

NVec_∞ ??

- Objects: normed vector spaces (real, say)
- Morphisms: all linear maps

Certainly an uninteresting category!

But becoming interesting when taken with its operator norms, here on a logarithmic scale:

$$|X \xrightarrow{f} Y| := \sup_{x \neq 0} \log^\circ\left(\frac{\|fx\|}{\|x\|}\right) \qquad (\text{with } \log^\circ \alpha = \max\{0, \log \alpha\})$$

$$|f| \in \mathcal{R}_+ = ([0,\infty], \geq, +, 0) \qquad 0 \geq |\mathrm{id}_X| \qquad |g| + |f| \geq |g \cdot f|$$

- $|f| < \infty \iff f$ bounded (continuous)
- $|f| = 0 \iff f$ 1-Lipschitz (non-expanding)

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Replacing $\mathcal{R}_+ = ([0,\infty], \geq, +, 0)$ by $\mathcal{V} = (\mathcal{V}, \leq, \otimes, \mathrm{k})$

In this talk, a *quantale* (always unital and commutative) is given by

- a complete lattice $(\mathcal{V}, \leq)$
- a commutative monoid $(\mathcal{V}, \otimes, \mathrm{k})$
- satisfying the infinite distributive law $u \otimes \bigvee_i v_i = \bigvee_i u \otimes v_i$

That is:

$\mathcal{V} = (\mathcal{V}, \leq, \otimes, \mathrm{k})$ is a small, thin, skeletal, cocomplete symmetric monoidal-closed category

Internal hom: $u \leq [v, w] \iff u \otimes v \leq w$

Key examples in this talk: $\mathcal{R}_+$ (Lawvere), $2 = (\{\mathrm{true}, \mathrm{false}\}, \Rightarrow, \wedge, \mathrm{true})$ (Boole)

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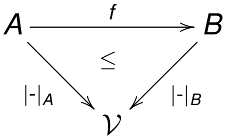
$\mathsf{Set}/\!/\mathcal{V} := \mathrm{Fam}(\mathcal{V}, \leq)$

$\mathcal{V}$-normed sets and maps:



$$|a|_A \leq |fa|_B$$

Set$/\!/\mathcal{V}$ is

- topological over Set
- locally presentable
- symmetric monoidal closed

$$A \otimes B = A \times B, \quad |(a,b)| = |a| \otimes |b| \qquad \mathrm{E} = \{*\}, \quad |*| = \mathrm{k}$$

$$[A,B] = \mathsf{Set}(A,B), \quad |\varphi| = \bigwedge_{a \in A} [|a|, |\varphi a|] \quad \text{(i.e. } |\varphi| \text{ is maximal with } |\varphi| \otimes |a| \leq |\varphi a|)$$

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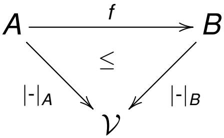

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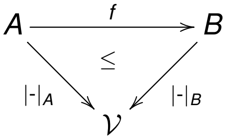

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Enrichment in Set$//\mathcal{V}$ defines $\mathcal{V}$-normed categories and their functors:

$|f|_{\mathbb{X}} \leq |Ff|_{\mathbb{Y}}$

$\mathrm{E} \to \mathbb{X}(x, x)$ $\mathbb{X}(x, y) \otimes \mathbb{X}(y, z) \to \mathbb{X}(x, z)$

$\mathrm{k} \leq |1_x|$ $|f| \otimes |g| \leq |g \cdot f|$

$\mathsf{Cat}/\!/\mathcal{V} := (\mathsf{Set}/\!/\mathcal{V})\text{-}\mathsf{Cat}, \quad \mathsf{CAT}/\!/\mathcal{V} := (\mathsf{Set}/\!/\mathcal{V})\text{-}\mathsf{CAT}$

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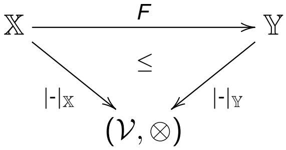

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$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\quad F \quad} & \mathbb{Y} \\ & \leq & \\ {\scriptstyle |\text{-}|_{\mathbb{X}}} \searrow & & \swarrow {\scriptstyle |\text{-}|_{\mathbb{Y}}} \\ & (\mathcal{V}, \otimes) & \end{array} \qquad |f|_{\mathbb{X}} \leq |Ff|_{\mathbb{Y}}$$

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$\mathcal{V}$-categories *vs.* (Set//$\mathcal{V}$)-categories

$$\mathcal{V} \underset{\text{s}}{\overset{\text{i}}{\underset{\top}{\rightleftarrows}}} \mathsf{Set}/\!/\mathcal{V} \;= \mathrm{Fam}(\mathcal{V}, \leq)$$

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$$\mathsf{Met}_1 \underset{\text{s}}{\overset{\text{i}}{\underset{\top}{\rightleftarrows}}} \mathsf{NCat}_1$$

$\mathcal{V} = 2$:

$$\mathsf{Ord} \underset{\text{s}}{\overset{\text{i}}{\underset{\top}{\rightleftarrows}}} \mathsf{Cat}/\!/2 \qquad \ni (\mathbb{X}, \mathcal{S}), \ \mathrm{Id}(\mathbb{X}) \subseteq \mathcal{S}, \ \mathcal{S} \cdot \mathcal{S} \subseteq \mathcal{S}$$

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Properties of $\mathsf{Cat}//\mathcal{V}$, $\mathcal{V}$-normed categories *vs.* ordinary categories

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- topological over Cat
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$$\mathbb{X} \otimes \mathbb{Y} = \mathbb{X} \times \mathbb{Y}, \quad |(f, f')| = |f| \otimes |f'| \qquad \mathbb{E} = \{* \to *\}, \quad |* \to *| = \mathrm{k}$$

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$$(v \mapsto \mathrm{true}) :\Longleftrightarrow \mathrm{k} \le v \qquad (\mathbb{X}, |\text{-}|) \longmapsto (\mathbb{X}, \mathbb{X}_\circ)$$

with $\mathbb{X}_\circ := \{f : \mathrm{k} \le |f|\}$ (à la Kelly)

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Every monoidal-closed category $\mathcal{W}$ becomes $\mathcal{W}$-enriched, *qua* its internal hom.

Q: What happens to $\mathcal{W} =$ Set$//\mathcal{V}$?

A: Obtain a category with $\mathcal{V}$-normed sets as objects and arbitrary maps as morphisms:

$$\mathsf{Set}||\mathcal{V}$$

Therefore:

Set$||\mathcal{V}$ is a $\mathcal{V}$-normed category, with $|\varphi : A \to B| = \bigwedge_{a \in A}[|a|, |\varphi a|]$. Furthermore:

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Normed convergence of sequences

$s : \mathbb{N} \longrightarrow \mathbb{X}$ $\quad x_o \xrightarrow{s_{0,1}} x_1 \xrightarrow{s_{1,2}} x_2 \longrightarrow \ldots \longrightarrow x_m \xrightarrow{s_{m,n}} x_n \longrightarrow \ldots$

$s_{|N} = s|_{\uparrow N}$ $\quad$ with $\uparrow N = \{n : n \geq N\}$

$x \cong \mathrm{ncolim}\, s$: (C1) $x \cong \mathrm{colim}\, s$ (in the ordinary category $\mathbb{X}$ with cocone $\gamma_n : x_n \to x$)

(C2) $\forall y \in \mathbb{X} : (\mathrm{Nat}(s_{|N}, \Delta y) \to \mathbb{X}(x, y))_{N \in \mathbb{N}}$ is a colimit cocone in $\mathrm{Set}//\mathcal{V}$

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Existence granted,
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A word about symmetry

$$(\mathrm{S}) \qquad |f \cdot h| \otimes |h| \leq |f|$$

$$(\mathrm{S}^{\mathrm{op}}) \qquad |g \cdot f| \otimes |g| \leq |f|$$

For $\mathbb{X} = \mathrm{i}X,\ X \in \mathcal{V}$-Cat:

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For $(\mathbb{X}, \mathcal{S}) \in \mathrm{Cat}//2$:

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But here we may not trade (S) for $(\mathrm{S}^{\mathrm{op}})$ (as done in [Kubiś 2017])!

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For $\mathbb{X} \in$ Cat//$\mathcal{V}$:

If $\mathbb{X}$ satisfies (S), then $(\mathrm{C1})$ & $(\mathrm{C2a})$ suffice to also have $(\mathrm{C2b})$.
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A word about symmetry

$$(\mathrm{S}) \qquad |f \cdot h| \otimes |h| \leq |f|$$

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Cauchy cocompleteness

$s : \mathbb{N} \to \mathbb{X}$ Cauchy $\quad :\iff \mathrm{k} \leq \bigvee_N \bigwedge_{n \geq N} |s_{m,n}|$

$\mathbb{X}$ Cauchy cocomplete $:\iff$ every Cauchy sequence in $\mathbb{X}$ has a normed colimit in $\mathbb{X}$

Caution: A sequence with a normed colimit may not be Cauchy!

For $\mathbb{X} = \mathrm{i}X,\ X \in \mathcal{R}_+\text{-Cat} = \mathrm{Met}_1$:

$$s \text{ Cauchy} \iff \inf_N \sup_{n \geq m \geq N} X(x_m, x_n) = 0$$

$$\iff s \text{ is forward Cauchy (see [Bonsangue, van Breugel, Rutten 1998])}$$

$$x \cong \mathrm{ncolim}\, s \iff \forall y :\ X(x, y) = \inf_N \sup_{n \geq N} X(x_n, y)$$

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Cauchy cocompleteness *vs.* idempotent completeness

Denote by $\overline{e}$ the constant sequence $x \xrightarrow{e} x \xrightarrow{e} x \longrightarrow \ldots$ given by an idempotent e.

$$\overline{e} \text{ has (ordinary) colimit in } \mathbb{X} \iff e \text{ splits } (e = t \cdot r,\ r \cdot t = 1)$$
$$\overline{e} \text{ is Cauchy} \iff \mathrm{k} \leq |e|$$
$$\overline{e} \text{ has normed colimit in } \mathbb{X} \iff e \text{ splits such that } \mathrm{k} \leq |r|,\ \mathrm{k} \leq |t|$$

Equivalent are for a $\mathcal{V}$-normed category $\mathbb{X}$:

- $\mathbb{X}_\circ$ is idempotent complete (*i.e.*, every idempotent in $\mathbb{X}_\circ$ splits in $\mathbb{X}_\circ$).
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Returning to normed vector spaces

For a constant $c > 0$, let $\mathbb{R}_c$ be the 1-dimensional vector space $\mathbb{R}$, normed by $|1|_c = c$.

Should we allow $c = 0$?

$$\mathbb{R} = \mathbb{R}_1 \longrightarrow \mathbb{R}_{\frac{1}{2}} \longrightarrow \mathbb{R}_{\frac{1}{3}} \longrightarrow \ldots \longrightarrow \mathbb{R}_0$$

Should we allow $c = \infty$?

$$\mathbb{R} = \mathbb{R}_1 \longleftarrow \mathbb{R}_2 \longleftarrow \mathbb{R}_3 \longleftarrow \ldots \longleftarrow \mathbb{R}_\infty$$

The price for “Yes”: Put $e^\infty = \infty$ and $0 \cdot \infty = \infty$ (!), then consider the adjunction

$$\mathcal{R}_+ = ([0,\infty], \geq, +, 0) \underset{\log^\circ}{\overset{\exp}{\underset{\bot}{\rightleftarrows}}} \mathcal{R}_\times = ([0,\infty], \geq, \cdot, 1)$$

Int'l hom $[\beta, \alpha]$: $\alpha \mathbin{\dot{-}} \beta = \max\{\alpha - \beta, 0\}$ $\qquad \frac{\alpha}{\beta} = \inf\{\gamma : \alpha \leq \beta \cdot \gamma\}$

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The normed category SNVec_∞

A seminorm $\|\text{-}\| : X \to [0,\infty]$ on a vector space X must satisfy:

- $\|0\| = 0$
- $\|ax\| = |a|\,\|x\|$ $(a \in \mathbb{R}, a \neq 0)$
- $\|x+y\| \leq \|x\| + \|y\|$

With all linear maps as morphisms and $|X \xrightarrow{f} Y| := \sup_{x\in X} \log^\circ(\frac{\|fx\|}{\|x\|})$, one obtains the normed category SNVec_∞.

Theorem

SNVec_∞ *is Cauchy cocomplete. But its full normed subcategory* NVec_∞ *is not.*

The proof is harder than one may have expected, although the starting point seems clear:

For a Cauchy sequence $s = (X_m \xrightarrow{s_{m,n}} X_n)_{m\leq n}$, form the colimit $(X_n \xrightarrow{\gamma_n} X)_n$ in Vec and put

$$\|x\| := \sup_{N\in\mathbb{N}} \inf_{n\geq N} \inf_{z\in\gamma_n^{-1}x} \|z\|_n \quad (x \in X)$$

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Some final remarks on NVec_∞

Call a linear map $f : X \to Y$ of seminormed vector spaces a 0-to-0 morphism if $\|x\| = 0$ alawys implies $\|fx\| = 0$. This defines the wide subcategory SNVec_{00} of SNVec_∞.

Corollary

The normed category NVec_∞ is a full reflective subcategory of SNVec_{00} (not of SNVec_∞). It has colimits of all those Cauchy sequences whose normed colimit in SNVec_∞ is also a colimit in the ordinary category SNVec_{00}.

An existing normed colimit in NVec_∞ of a Cauchy sequence of isometric embeddings of Banach spaces may not be Banach:

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No linear structure: Is Met_∞ Cauchy cocomplete?

Met_∞:

- Objects: (Lawvere) metric spaces
- Morphisms: all maps, normed by

$$|X \xrightarrow{\varphi} Y| := \sup_{x,x' \in X} \log^\circ\left(\frac{Y(\varphi x, \varphi x')}{X(x,x')}\right)$$

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Yes: Met_∞ is Cauchy cocomplete!

Recall "totally below":

$$u \ll \bigvee_{i \in I} v_i \iff \exists\, i \in I : u \ll v_I$$

Theorem

$\mathcal{V}$-Lip *is Cauchy cocomplete if*

- $\Downarrow \mathrm{k} = \{\varepsilon \in \mathcal{V} : \varepsilon \ll \mathrm{k}\}$ *is up-directed;*
- k *is approximated from totally below:* $\bigvee \Downarrow \mathrm{k} = \mathrm{k}$*;*
- $\otimes$ *preserves* $\ll$*:* $(u \ll v,\ w > \perp \Longrightarrow u \otimes w \ll v \otimes w)$.

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Theorem

For every small $\mathcal{V}$-normed category $\mathbb{X}$, the $\mathcal{V}$-normed presheaf category

$$[\mathbb{X}, \mathsf{Set}||\mathcal{V}]$$

is Cauchy cocomplete, provided that $\mathcal{V}$ satisfies

- (A) k *is approximated from totally below:* $\bigvee \Downarrow \mathrm{k} = \mathrm{k}$;

  OR

- (B) k $\wedge$*-distributes over arbitrary joins:* $\mathrm{k} \wedge \bigvee_i v_i = \bigvee_i \mathrm{k} \wedge v_i$.

The proof is much harder than expected!
Conditions (A) and (B) are independent of each other.
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Reminders: weighted colimits, distributors, accessible presheaves

$F : \mathbb{A} \to \mathbb{X}$, $\varphi : \mathbb{A}^{\mathrm{op}} \to \mathsf{Set}||\mathcal{V}$ $\mathcal{V}$-normed functors of $\mathcal{V}$-normed categories $\mathbb{A}, \mathbb{X}$ ($\mathbb{A}$ small),

also written as composable $\mathcal{V}$-distributors: $F^* : \mathbb{X} \longrightarrow\!\!\!\!\!\circ\;\; \mathbb{A}$, $\varphi : \mathbb{A} \longrightarrow\!\!\!\!\!\circ\;\; \mathbb{E}$

$$x \cong \mathrm{colim}^{\varphi} F \iff \mathbb{X}(x, y) \cong \mathrm{Nat}(\varphi, \mathbb{X}(F-, y)) \text{ naturally in } y$$

$$\iff x \cong \mathrm{colim}^{\varphi \cdot F^*} \mathrm{id}_{\mathbb{X}}$$

$$\iff: \text{“ } x \text{ is a weighted colimit of } \varphi \cdot F^* \text{ ”}$$

After [Kelly-Schmitt 2005]:

$\psi : \mathbb{X}^{\mathrm{op}} \to \mathsf{Set}||\mathcal{V}$ accessible $:\iff \psi = \varphi \cdot F^*$ for some F, φ as above

$\mathcal{P}\mathbb{X} :=$ full normed subcategory of $[\mathbb{X}^{\mathrm{op}}, \mathsf{Set}||\mathcal{V}]$ of all accessible presheaves on $\mathbb{X}$

Reminders: weighted colimits, distributors, accessible presheaves

$F : \mathbb{A} \to \mathbb{X}$, $\varphi : \mathbb{A}^{\mathrm{op}} \to \mathsf{Set}||\mathcal{V}$ $\mathcal{V}$-normed functors of $\mathcal{V}$-normed categories $\mathbb{A}, \mathbb{X}$ ($\mathbb{A}$ small),

also written as composable $\mathcal{V}$-distributors: $F^* : \mathbb{X} \longrightarrow\!\!\circ\!\!\rightarrow \mathbb{A}$, $\varphi : \mathbb{A} \longrightarrow\!\!\circ\!\!\rightarrow \mathbb{E}$

$$x \cong \mathrm{colim}^{\varphi} F \iff \mathbb{X}(x, y) \cong \mathrm{Nat}(\varphi, \mathbb{X}(F-, y)) \text{ naturally in } y$$
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Normed colimits as weighted colimits

Proposition

If $\mathcal{V}$ satisfies condition (A) or (B), then for every $\mathcal{V}$-normed category $\mathbb{X}$, $\mathcal{P}\mathbb{X}$ is Cauchy cocomplete.

For a Cauchy sequence s in the $\mathcal{V}$-normed category $\mathbb{X}$, form

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Proposition

$$x \cong \mathrm{ncolim}\, s \iff x \cong \mathrm{colim}^{\varphi_s}\mathrm{id}_{\mathbb{X}}$$

Corollary

$\mathbb{X}$ Cauchy cocomplete $\iff$ $\mathbb{X}$ has weighted colimits for all $F : \mathbb{A} \to \mathbb{X}$, $\varphi : \mathbb{A}^{\mathrm{op}} \to \mathsf{Set}||\mathcal{V}$, with $\mathbb{A}$ countable and φ a normed colimit of a Cauchy sequence of representables in $\mathcal{P}\mathbb{A}$.

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Cauchy cocompletion (à la [Kelly, Schmitt 2005])

$\mathcal{V}$ continues to satisfy (A) or (B).

Let Φ be the class of weights used in the Corollary, so that

$$\mathbb{X} \text{ is Cauchy cocomplete} \iff \mathbb{X} \text{ is } \Phi\text{-cocomplete}.$$

Let $\Phi(\mathbb{X})$ be the least full replete $\mathcal{V}$-normed subcategory of $\mathcal{P}\mathbb{X}$ closed under Φ-colimits.

Theorem

For every $\mathcal{V}$-normed category $\mathbb{X}$ and every Cauchy cocomplete $\mathcal{V}$-normed category $\mathbb{Y}$, the composition with the restricted Yoneda embedding $\mathbf{y}_{\mathbb{X}} : \mathbb{X} \to \Phi(\mathbb{X})$ defines an equivalence

$$(\Phi\text{-COCTS})(\Phi(\mathbb{X}), \mathbb{Y}) \to (\mathsf{CAT}//\mathcal{V})(\mathbb{X}, \mathbb{Y}).$$

That is, $\Phi(-)$ provides a left biadjoint to the inclusion 2-functor $\Phi\text{-COCTS} \to (\mathsf{CAT}//\mathcal{V})$.

The equivalence restricts to $(\Phi\text{-Cocts})(\Phi(\mathbb{X}), \mathbb{Y}) \to (\mathsf{Cat}//\mathcal{V})(\mathbb{X}, \mathbb{Y})$ for small $\mathbb{X}$ and $\mathbb{Y}$.

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Banach's Fixed Point Theorem

Let $\mathbb{X}$ be ($\mathcal{R}_+$-)normed and $F : \mathbb{X} \to \mathbb{X}$ contractive: there is $\ell < 1$ with $|Fh| \leq \ell|h|$ for all h.

Suppose we have some $f : x \to Fx$ with $|f| < \infty$. Just like for metric spaces, the sequence

$$s_f = (\ x \xrightarrow{f} Fx \xrightarrow{Ff} F^2x \xrightarrow{F^2f} F^3x \xrightarrow{F^3f} \dots\ )$$

is Cauchy. Would its colimit be a "fixed point" of F?

Theorem

Let $\mathbb{X}$ be Cauchy cocomplete with some f as above. If the contractive endofunctor

- *F preserves $y \cong \operatorname{colim} s_f$, then the canonical $\bar{f} : y \to Fy$ is an iso with $|\bar{f}| = 0$;*
- *F preserves $y \cong \operatorname{ncolim} s_f$, then the canonical $\bar{f} : y \to Fy$ is a 0-iso: $|\bar{f}| = 0 = |\bar{f}^{-1}|$.*

Note: Preservation of the normed colimit follows from its ordinary preservation when $\mathbb{X}$ satisfies the symmetry condition (S) or (S^{op}).

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A To-Do list

- Find a quantale $\mathcal{V}$ such that Set$||\mathcal{V}$ fails to be Cauchy cocomplete!
- Why not directed or filtered systems instead of just sequences? Relevant examples?
- Beyond quantales: $\mathcal{V}$ any symmetric monoidal-closed category, ... ?
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GRACIAS!