Lawvere Theories and Symmetric Operads as Substitution Algebras: Free constructions for Abstract Syntax

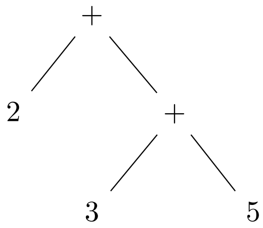
Sanjiv Ranchod

(joint work with Marcelo Fiore)

University of Cambridge

CT2024

$$2 + (3 + 5) \quad \rightsquigarrow$$



Second-Order Theories

A second-order signature is a pair (Ω, a) where Ω is a set (of operations) and $a : \Omega \to \mathbb{N}^*$ is an (arity) map.

Operation: $\omega \in \Omega$

Arity: $a(\omega) = (n_1, \ldots, n_k)$

Contexts

Contexts: $x_1, \ldots, x_n = \underline{\mathrm{n}}$

Contexts

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Cartesian Contexts

Weakening: $$\frac{\underline{n} \;\vdash\; t}{\underline{n+1} \;\vdash\; t}$$

Contraction: $$\frac{\underline{n+1} \;\vdash\; t}{\underline{n} \;\vdash\; t\{x_n/x_{n+1}\}}$$

Exchange: $$\frac{\underline{n+2} \;\vdash\; t}{\underline{n+2} \;\vdash\; t\{x_{n+2}/x_{n+1}, x_{n+1}/x_{n+2}\}}$$

Contexts

Contexts: $x_1, \ldots, x_n = \underline{\mathrm{n}}$

Linear Contexts

Exchange:
$$\frac{\underline{\mathrm{n}+2} \;\vdash\; t}{\underline{\mathrm{n}+2} \;\vdash\; t\{x_{n+2}/x_{n+1}, x_{n+1}/x_{n+2}\}}$$

Terms-in-context

Cartesian Terms:

$$\frac{1 \leq i \leq n}{\underline{\mathrm{n}} \vdash x_i}$$

$$\frac{\underline{\underline{\mathrm{n}+\mathrm{n}_1}} \vdash t_1 \quad \dots \quad \underline{\underline{\mathrm{n}+\mathrm{n}_\mathrm{k}}} \vdash t_k}{\underline{\mathrm{n}} \vdash \omega(t_1, \dots, t_k)} \; a(\omega) = (n_1, \dots, n_k)$$

Terms-in-context

Cartesian Terms:

$$\frac{1 \leq i \leq n}{\underline{\mathrm{n}} \ \vdash \ x_i}$$

$$\frac{\underline{\underline{\mathrm{n}+\mathrm{n_1}}} \ \vdash \ t_1 \quad \ldots \quad \underline{\underline{\mathrm{n}+\mathrm{n_k}}} \ \vdash \ t_k}{\underline{\mathrm{n}} \ \vdash \omega(t_1, \ldots, t_k)} \ a(\omega) = (n_1, \ldots, n_k)$$

Linear Terms:

$$\frac{}{\underline{1} \ \vdash \ x_1}$$

$$\frac{\underline{\mathrm{m_1}} \ \vdash \ t_1 \quad \ldots \quad \underline{\mathrm{m_k}} \ \vdash \ t_k}{\pi(\underline{\mathrm{m_1} + \ldots \mathrm{m_k}}) \ \vdash \omega(t_1, \ldots, t_k)} \ a(\omega) = (n_1, \ldots, n_k)$$

Single-variables substitution

Cartesian Substitution:

$$\frac{\underline{\underline{\mathrm{n}+1}} \vdash t \qquad \underline{\mathrm{n}} \vdash u}{\underline{\mathrm{n}} \vdash t\{u/x_{n+1}\}}$$

Linear Substitution:

$$\frac{\underline{\underline{\mathrm{n}+1}} \vdash t \qquad \underline{\mathrm{m}} \vdash u}{\underline{\mathrm{n}+\mathrm{m}} \vdash t\{u/x_{n+1}\}}$$

The Category of Contexts

Category of Contexts:
$(\mathbb{C}, \otimes, I, b)$

Category for models:
$\mathcal{C} = \mathbf{Set}^{\mathbb{C}}$

For $\mathbb{C}$:

Objects:	contexts
Morphisms:	context renamings
Tensor $\otimes$:	pairings
Monoidal unit I:	empty context
Symmetry b	exchange operation

Each $A \in \mathbb{C}$ induces the endofunctor $- \otimes A : \mathbb{C} \to \mathbb{C}$

Symmetric Monads and Endofunctors

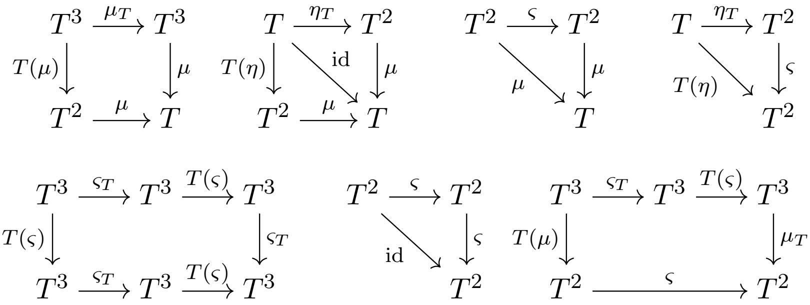
A symmetric monoid in a (non-symmetric) monoidal category is a quadruple (M, c, w, s) where M is an object and $c : M \otimes M \to M$, $w : I \to M$ and $s : M \otimes M \to M \otimes M$ are morphisms with

$$\begin{array}{ccc} M^3 & \xrightarrow{c\otimes \mathrm{id}} & M^2 \\ {\scriptstyle \mathrm{id}\otimes c}\downarrow & & \downarrow{\scriptstyle c} \\ M^2 & \xrightarrow{c} & M \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\mathrm{id}\otimes w} & M^2 \\ {\scriptstyle w\otimes \mathrm{id}}\downarrow & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle c} \\ M^2 & \xrightarrow{c} & M \end{array} \qquad \begin{array}{ccc} M^2 & \xrightarrow{s} & M^2 \\ & {}_{c}\searrow & \downarrow{\scriptstyle c} \\ & & M \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{w\otimes \mathrm{id}} & M^2 \\ & {}_{\mathrm{id}\otimes w}\searrow & \downarrow{\scriptstyle s} \\ & & M^2 \end{array}$$

$$\begin{array}{ccccc} M^3 & \xrightarrow{s\otimes \mathrm{id}} & M^3 & \xrightarrow{\mathrm{id}\otimes s} & M^3 \\ {\scriptstyle \mathrm{id}\otimes s}\downarrow & & & & \downarrow{\scriptstyle s\otimes \mathrm{id}} \\ M^3 & \xrightarrow{s\otimes \mathrm{id}} & M^3 & \xrightarrow{\mathrm{id}\otimes s} & M^3 \end{array} \qquad \begin{array}{ccc} M^2 & \xrightarrow{s} & M^2 \\ & {}_{\mathrm{id}}\searrow & \downarrow{\scriptstyle s} \\ & & M^2 \end{array} \qquad \begin{array}{ccccc} M^3 & \xrightarrow{s\otimes \mathrm{id}} & M^3 & \xrightarrow{\mathrm{id}\otimes s} & M^3 \\ {\scriptstyle \mathrm{id}\otimes c}\downarrow & & & & \downarrow{\scriptstyle c\otimes \mathrm{id}} \\ M^2 & & \xrightarrow{\qquad s \qquad} & & M^2 \end{array}$$

Symmetric Monads and Endofunctors

A symmetric monad is a quadruple $(T, \mu, \eta, \varsigma)$ where T is an endofunctor and $\mu : T^2 \to T$, $\eta : \mathrm{id} \to T$ and $\varsigma : T^2 \to T^2$ are natural transformations with



Symmetric Monads and Endofunctors

A symmetric endofunctor is a pair (T, ς) where T is an endofunctor and $\varsigma : T^2 \to T^2$ is a natural transformation with

$$\begin{array}{ccccc} T^3 & \xrightarrow{\varsigma_T} & T^3 & \xrightarrow{T(\varsigma)} & T^3 \\ {\scriptstyle T(\varsigma)}\big\downarrow & & & & \big\downarrow{\scriptstyle \varsigma_T} \\ T^3 & \xrightarrow{\varsigma_T} & T^3 & \xrightarrow{T(\varsigma)} & T^3 \end{array} \qquad \begin{array}{ccc} T^2 & \xrightarrow{\varsigma} & T^2 \\ & {\scriptstyle \mathrm{id}}\searrow & \big\downarrow{\scriptstyle \varsigma} \\ & & T^2 \end{array}$$

Category for models

Category for Models: $\mathcal{C} = \mathbf{Set}^{\mathbb{C}}$

Objects: $X(A)$ is the set of terms for X in context A

$\mathcal{C}$ is a topos $\implies$ $\mathcal{C}$ complete and cocomplete

$(\mathcal{C}, \times, 1)$ cartesian monoidal $\quad (\mathcal{C}, +, 0)$ cocartesian monoidal

$\mathcal{C}$ is cartesian closed with $- \times X \dashv (-)^X$

Day Convolution

The Day convolution is induced as the left Kan extension of $\mathcal{Y}(- \otimes -) : \mathbb{C} \times \mathbb{C} \to \mathcal{C}$ along $\mathcal{Y} \times \mathcal{Y}$ given by the coend formula

$$(X \hat{\otimes} Y)(A) = \int^{B_1, B_2 \in \mathbb{C}} \mathbb{C}(B_1 \otimes B_2, A) \times X(B_1) \times Y(B_2)$$

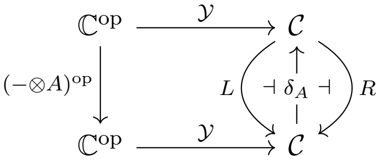
Monoidal unit: $\mathcal{Y}(I)$

Linear exponential: $- \hat{\otimes} X \dashv X \multimap -$

Models pairings of presheafs

Context Extension δ

For each context $A \in \mathbb{C}$:



$$\delta_A(X)(B) = X(B \otimes A)$$

δ_A is a linear exponential: $L = -\hat{\otimes}\mathcal{Y}(A) \implies \delta_A \cong \mathcal{Y}(A) \multimap -$

δ_A monoidal for $\times$ and $+$, but not $\hat{\otimes}$

δ_A is a symmetric endofunctor

Strong Symmetric Monads

A strong endofunctor F has

$$\mathbf{str}_{A,B} : F(A) \otimes B \to F(A \otimes B)$$

Strong Symmetric Monads

A strong monad (T, μ, η) has

$$\mathbf{str}_{A,B} : T(A) \otimes B \to T(A \otimes B)$$

with

$$\begin{array}{ccccc}
T^2(A) \otimes B & \xrightarrow{\mathbf{str}} & T(T(A) \otimes B) & \xrightarrow{T(\mathbf{str})} & T^2(A \otimes B) \\
\downarrow{\scriptstyle \mu \otimes \mathrm{id}} & & & & \downarrow{\scriptstyle \mu} \\
T(A) \otimes B & & \xrightarrow[\mathbf{str}]{\qquad\qquad} & & T(A \otimes B)
\end{array}
\qquad
\begin{array}{ccc}
A \otimes B & & \\
\downarrow{\scriptstyle \eta \otimes \mathrm{id}} & \searrow{\scriptstyle \eta} & \\
T(A) \otimes B & \xrightarrow[\mathbf{str}]{} & T(A \otimes B)
\end{array}$$

Strong Symmetric Monads

A strong symmetric monad $(T, \mu, \eta, \varsigma)$ has

$$\mathbf{str}_{A,B} : T(A) \otimes B \to T(A \otimes B)$$

with

$$\begin{array}{ccccc}
T^2(A) \otimes B & \xrightarrow{\mathbf{str}} & T(T(A) \otimes B) & \xrightarrow{T(\mathbf{str})} & T^2(A \otimes B) \\
\downarrow{\scriptstyle \mu \otimes \mathrm{id}} & & & & \downarrow{\scriptstyle \mu} \\
T(A) \otimes B & & \xrightarrow[\mathbf{str}]{} & & T(A \otimes B)
\end{array}$$

$$\begin{array}{ccc}
A \otimes B & & \\
\downarrow{\scriptstyle \eta \otimes \mathrm{id}} & \searrow{\scriptstyle \eta} & \\
T(A) \otimes B & \xrightarrow[\mathbf{str}]{} & T(A \otimes B)
\end{array}$$

$$\begin{array}{ccccc}
T^2(A) \otimes B & \xrightarrow{\mathbf{str}} & T(T(A) \otimes B) & \xrightarrow{T(\mathbf{str})} & T^2(A \otimes B) \\
\downarrow{\scriptstyle \varsigma \otimes \mathrm{id}} & & & & \downarrow{\scriptstyle \varsigma} \\
T^2(A) \otimes B & \xrightarrow[\mathbf{str}]{} & T(T(A) \otimes B) & \xrightarrow[T(\mathbf{str})]{} & T^2(A \otimes B)
\end{array}$$

Strong Symmetric Monads

A strong symmetric endofunctor (T, ς) has

$$\mathbf{str}_{A,B} : T(A) \otimes B \to T(A \otimes B)$$

with

$$\begin{array}{ccccc}
T^2(A) \otimes B & \xrightarrow{\mathbf{str}} & T(T(A) \otimes B) & \xrightarrow{T(\mathbf{str})} & T^2(A \otimes B) \\
{\scriptstyle \varsigma \otimes \mathrm{id}} \downarrow & & & & \downarrow {\scriptstyle \varsigma} \\
T^2(A) \otimes B & \xrightarrow[\mathbf{str}]{} & T(T(A) \otimes B) & \xrightarrow[T(\mathbf{str})]{} & T^2(A \otimes B)
\end{array}$$

Symmetric Distributive Laws

A distributive law between an endofunctor S and a monad (T, μ, η) is a natural transformation

$$\tau : TS \to ST$$

with

$$\begin{array}{ccccc} T^2S & \xrightarrow{T(\tau)} & TST & \xrightarrow{\tau_T} & ST^2 \\ \downarrow{\scriptstyle \mu_S} & & & & \downarrow{\scriptstyle S(\mu)} \\ TS & & \xrightarrow[\tau]{\hspace{4em}} & & ST \end{array} \qquad \begin{array}{ccc} S & & \\ \downarrow{\scriptstyle \eta_S} & \searrow^{S(\eta)} & \\ TS & \xrightarrow[\tau]{} & ST \end{array}$$

Symmetric Distributive Laws

A symmetric distributive law between an endofunctor S and a symmetric monad $(T, \mu, \eta, \varsigma)$ is a natural transformation

$$\tau : TS \to ST$$

with

$$\begin{array}{ccccc} T^2S & \xrightarrow{T(\tau)} & TST & \xrightarrow{\tau_T} & ST^2 \\ \downarrow{\scriptstyle \mu_S} & & & & \downarrow{\scriptstyle S(\mu)} \\ TS & & \xrightarrow[\tau]{\hspace{6em}} & & ST \end{array} \qquad \begin{array}{ccc} S & & \\ \downarrow{\scriptstyle \eta_S} & \searrow^{S(\eta)} & \\ TS & \xrightarrow[\tau]{} & ST \end{array}$$

$$\begin{array}{ccccc} T^2S & \xrightarrow{T(\tau)} & TST & \xrightarrow{\tau_T} & ST^2 \\ \downarrow{\scriptstyle \varsigma_S} & & & & \downarrow{\scriptstyle S(\tau)} \\ TS & \xrightarrow[T(\tau)]{} & TST & \xrightarrow[\tau_T]{} & ST^2 \end{array}$$

Symmetric Distributive Laws

A symmetric transformation between an endofunctor S and a symmetric endofunctor (T, ς) is a natural transformation

$$\tau : TS \to ST$$

with

$$\begin{array}{ccccc}
T^2S & \xrightarrow{T(\tau)} & TST & \xrightarrow{\tau_T} & ST^2 \\
\Big\downarrow{\scriptstyle \varsigma_S} & & & & \Big\downarrow{\scriptstyle S(\tau)} \\
TS & \xrightarrow[T(\tau)]{} & TST & \xrightarrow[\tau_T]{} & ST^2
\end{array}$$

Generalised Parameterised Structural Recursion

For endofunctors F and F' and initial F-algebra $\alpha : F(A) \to A$ and for some endofunctor G with right adjoint , and natural transformation $\psi : GF \to F'G$, there exists a unique k, with

$$
\begin{array}{ccc}
F'G(A) & \xrightarrow{F'(k)} & F'(B) \\
\uparrow{\scriptstyle \psi_A} & & \\
GF(A) & & \downarrow{\scriptstyle \beta} \\
\downarrow{\scriptstyle G(\alpha)} & & \\
G(A) & \xrightarrow{k} & B
\end{array}
$$

The Cartesian Case

Category of Contexts: $\mathbb{F} \cong \mathrm{Skel}(\mathbf{FinSet})$

Objects: $\mathbf{n} = \{1, \ldots, n\}$

Morphisms: all maps

$\mathbb{F}$ is strict cocartesian : $\mathbf{n} \xrightarrow{\mathrm{old}_{\mathbf{n}}} \mathbf{n+1} \xleftarrow{\mathrm{new}_{\mathbf{n}}} \mathbf{1}$

Atomic morphism:

$$s = [\mathrm{new}_{\mathbf{1}}, \mathrm{old}_{\mathbf{1}}] : \mathbf{2} \to \mathbf{2}$$
$$w = \mathrm{old}_{\mathbf{0}} : \mathbf{0} \to \mathbf{1}$$
$$c = [\mathrm{id}_{\mathbf{1}}, \mathrm{id}_{\mathbf{1}}] : \mathbf{2} \to \mathbf{1}$$

The Cartesian Case

Category for models: *Object classifier topos:* $\mathcal{F} = \mathbf{Set}^{\mathbb{F}}$

Day convolution is cartesian: $\hat{\otimes} = \times$

δ_1 is a symmetric monad written $(\delta, \mathrm{up}, \mathrm{cont}, \mathrm{swap})$

Signature endofunctor:

$$\Sigma(X) = \coprod_{\omega \in \Omega} \prod_{n_i \in a(\omega)} \delta^{n_i}(X) : \mathcal{F} \to \mathcal{F}$$

Σ is strong: $\mathbf{str} : \Sigma(X) \times Y \to \Sigma(X \times Y)$

Σ has swapping: $\mathbf{swap} : \delta\Sigma(X) \xrightarrow{\cong} \Sigma\delta(X)$

The Cartesian Case

Σ algebra: $\alpha : \Sigma(X) \to X$

$$\alpha : \Sigma(X) \to X \quad \longmapsto \quad X$$

$$\Sigma\text{−}\mathbf{Alg} \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{F} \qquad (F \dashv U)$$

$$\varphi : \Sigma(TX) \to TX \quad \hookleftarrow \quad X$$

where $[\eta, \varphi] : X + \Sigma(TX) \to TX$ intial $X + \Sigma(-)$-algebra

The Cartesian Case

Σ algebra: $\alpha : \Sigma(X) \to X$

$$\alpha : \Sigma(X) \to X \quad \longmapsto \quad X$$

$$\Sigma\text{–}\mathbf{Alg} \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{F} \quad (U \dashv F)$$

$$\varphi : \Sigma(TX) \to TX \quad \hookleftarrow \quad X$$

where $[\eta, \varphi] : X + \Sigma(TX) \to TX$ intial $X + \Sigma(-)$-algebra

Presheaf of variables: $V = \mathcal{Y}(\mathbf{1})$

Model of Abstract Syntax: $\varphi : \Sigma(TV) \to TV$

Subsitution Algebra

A substitution algebra is a triple (X, σ, ν) where $X \in \mathcal{F}$, $\sigma : \delta(X) \times X \to X$ and $\nu : 1 \to \delta(X)$ with

$$
\begin{array}{ccc}
1 \times X & \xrightarrow{\pi_2} & X \\
{\scriptstyle \nu \times \mathrm{id}} \downarrow & \nearrow {\scriptstyle \sigma} & \\
\Sigma_{\mathrm{s}}(X) & &
\end{array}
\qquad
\begin{array}{ccc}
X \times X & \xrightarrow{\pi_1} & X \\
{\scriptstyle \mathrm{up}_X \times \mathrm{id}} \downarrow & \nearrow {\scriptstyle \sigma} & \\
\Sigma_{\mathrm{s}}(X) & &
\end{array}
\qquad
\begin{array}{ccccc}
\delta^2(X) \times 1 & & \xrightarrow{\pi_1} & & \delta^2(X) \\
\downarrow {\scriptstyle \mathrm{id} \times \nu} & & & & {\scriptstyle \mathrm{cont}} \downarrow \\
\delta^2(X) \times \delta(X) & \xrightarrow{\cong} & \delta\Sigma_{\mathrm{s}}(X) & \xrightarrow{\delta(\sigma)} & \delta(X)
\end{array}
$$

$$
\begin{array}{ccccccc}
\delta\Sigma_{\mathrm{s}}(X) \times X & \xrightarrow{\cong} & \Sigma_{\mathrm{s}}\delta(X) \times X & \xrightarrow{\mathbf{str}} & \Sigma_{\mathrm{s}}\Sigma_{\mathrm{s}}(X) & \xrightarrow{\Sigma_{\mathrm{s}}(\sigma)} & \Sigma_{\mathrm{s}}(X) \\
{\scriptstyle \delta(\sigma) \times \mathrm{id}} \downarrow & & & & & & \downarrow {\scriptstyle \sigma} \\
\Sigma_{\mathrm{s}}(X) & & & \xrightarrow{\sigma} & & & X
\end{array}
$$

where $\Sigma_s(X) = \delta(X) \times X$

Subsitution Algebra

A substitution algebra is a triple (X, σ, ν) where $X \in \mathcal{F}$, $\sigma : \delta(X) \times X \to X$ and $\nu : 1 \to \delta(X)$ with

$$
\begin{array}{ccc}
1 \times X & \xrightarrow{\pi_2} & X \\
{\scriptstyle \nu \times \mathrm{id}} \downarrow & \nearrow {\scriptstyle \sigma} & \\
\Sigma_{\mathrm{s}}(X) & &
\end{array}
\qquad
\begin{array}{ccc}
X \times X & \xrightarrow{\pi_1} & X \\
{\scriptstyle \mathrm{up}_X \times \mathrm{id}} \downarrow & \nearrow {\scriptstyle \sigma} & \\
\Sigma_{\mathrm{s}}(X) & &
\end{array}
\qquad
\begin{array}{ccccc}
\delta^2(X) \times 1 & & \xrightarrow{\pi_1} & & \delta^2(X) \\
\downarrow {\scriptstyle \mathrm{id} \times \nu} & & & & {\scriptstyle \mathrm{cont}} \downarrow \\
\delta^2(X) \times \delta(X) & \xrightarrow{\cong} & \delta\Sigma_{\mathrm{s}}(X) & \xrightarrow{\delta(\sigma)} & \delta(X)
\end{array}
$$

$$
\begin{array}{ccccccc}
\delta\Sigma_{\mathrm{s}}(X) \times X & \xrightarrow{\cong} & \Sigma_{\mathrm{s}}\delta(X) \times X & \xrightarrow{\mathbf{str}} & \Sigma_{\mathrm{s}}\Sigma_{\mathrm{s}}(X) & \xrightarrow{\Sigma_{\mathrm{s}}(\sigma)} & \Sigma_{\mathrm{s}}(X) \\
{\scriptstyle \delta(\sigma) \times \mathrm{id}} \downarrow & & & & & & \downarrow {\scriptstyle \sigma} \\
\Sigma_{\mathrm{s}}(X) & & & \xrightarrow{\sigma} & & & X
\end{array}
$$

where $\Sigma_s(X) = \delta(X) \times X$

SubstAlg $\cong$ Law

Subsitution Algebra

A substitution algebra is a triple (X, σ, ν) where $X \in \mathcal{F}$, $\sigma : \delta(X) \times X \to X$ and $\nu : 1 \to \delta(X)$ with

$$
\begin{array}{ccc}
1 \times X & \xrightarrow{\pi_2} & X \\
{\scriptstyle \nu \times \mathrm{id}} \downarrow & \nearrow {\scriptstyle \sigma} & \\
\Sigma_{\mathrm{s}}(X) & &
\end{array}
\qquad
\begin{array}{ccc}
X \times X & \xrightarrow{\pi_1} & X \\
{\scriptstyle \mathrm{up}_X \times \mathrm{id}} \downarrow & \nearrow {\scriptstyle \sigma} & \\
\Sigma_{\mathrm{s}}(X) & &
\end{array}
\qquad
\begin{array}{ccccc}
\delta^2(X) \times 1 & & \xrightarrow{\pi_1} & & \delta^2(X) \\
\downarrow {\scriptstyle \mathrm{id} \times \nu} & & & & {\scriptstyle \mathrm{cont}} \downarrow \\
\delta^2(X) \times \delta(X) & \xrightarrow{\cong} & \delta\Sigma_{\mathrm{s}}(X) & \xrightarrow{\delta(\sigma)} & \delta(X)
\end{array}
$$

$$
\begin{array}{ccccccc}
\delta\Sigma_{\mathrm{s}}(X) \times X & \xrightarrow{\cong} & \Sigma_{\mathrm{s}}\delta(X) \times X & \xrightarrow{\mathbf{str}} & \Sigma_{\mathrm{s}}\Sigma_{\mathrm{s}}(X) & \xrightarrow{\Sigma_{\mathrm{s}}(\sigma)} & \Sigma_{\mathrm{s}}(X) \\
{\scriptstyle \delta(\sigma) \times \mathrm{id}} \downarrow & & & & & & \downarrow {\scriptstyle \sigma} \\
\Sigma_{\mathrm{s}}(X) & & & \xrightarrow{\sigma} & & & X
\end{array}
$$

where $\Sigma_s(X) = \delta(X) \times X$

$$\mathbf{SubstAlg} \cong \mathbf{Law}$$

$\varphi : \Sigma(TV) \to TV$ has induced substitution algebra

The Linear Case

Category of Contexts: $\mathbb{B} \cong \mathrm{Core}(\mathbf{Finset})$

Objects: $\mathbf{n} = \{1, \ldots, n\}$

Morphisms: all bijections

$\mathbb{B}$ is symmetric monoidal

Category for models: *Combinatorial Species:* $\mathcal{B} = \mathbf{Set}^{\mathbb{B}}$

δ_1 is a symmetric endofunctor written (δ, swap)

Signature endofunctor:

$$\Sigma(X) = \coprod_{\omega \in \Omega} \widehat{\bigotimes_{n_i \in a(\omega)}} \delta^{n_i}(X)$$

Model for Abstract Syntax: $\varphi : \Sigma(TV) \to TV \quad$ for $V = \mathcal{Y}(1)$

The Linear Case

Leibniz isomorphism: $\delta(A\hat{\otimes}B) \cong \delta(A)\hat{\otimes}B + A\hat{\otimes}\delta(B)$

Derived endofunctor: $\Sigma' : \mathcal{B}^2 \to \mathcal{B}$

The Linear Case

Leibniz isomorphism: $\delta(A \hat{\otimes} B) \cong \delta(A) \hat{\otimes} B + A \hat{\otimes} \delta(B)$

Derived endofunctor: $\Sigma' : \mathcal{B}^2 \to \mathcal{B}$

$$\delta(X) \hat{\otimes} X$$

The Linear Case

Leibniz isomorphism: $\delta(A \hat{\otimes} B) \cong \delta(A) \hat{\otimes} B + A \hat{\otimes} \delta(B)$

Derived endofunctor: $\Sigma' : \mathcal{B}^2 \to \mathcal{B}$

$$\delta(\delta(X) \hat{\otimes} X)$$

The Linear Case

Leibniz isomorphism: $\delta(A \hat{\otimes} B) \cong \delta(A) \hat{\otimes} B + A \hat{\otimes} \delta(B)$

Derived endofunctor: $\Sigma' : \mathcal{B}^2 \to \mathcal{B}$

$$\delta\delta(X) \hat{\otimes} X + \delta(X) \hat{\otimes} \delta(X)$$

The Linear Case

Leibniz isomorphism: $\delta(A \hat{\otimes} B) \cong \delta(A) \hat{\otimes} B + A \hat{\otimes} \delta(B)$

Derived endofunctor: $\Sigma' : \mathcal{B}^2 \to \mathcal{B}$

$$\delta\delta(X) \hat{\otimes} X + \delta(X) \hat{\otimes} \delta(X)$$

The Linear Case

Leibniz isomorphism: $\delta(A \hat{\otimes} B) \cong \delta(A) \hat{\otimes} B + A \hat{\otimes} \delta(B)$

Derived endofunctor: $\Sigma' : \mathcal{B}^2 \to \mathcal{B}$

$$\delta(Y) \hat{\otimes} X + \delta(X) \hat{\otimes} Y$$

The Linear Case

Leibniz isomorphism: $\delta(A\hat{\otimes}B) \cong \delta(A)\hat{\otimes}B + A\hat{\otimes}\delta(B)$

Derived endofunctor: $\Sigma' : \mathcal{B}^2 \to \mathcal{B}$

$$\delta(Y)\hat{\otimes}X + \delta(X)\hat{\otimes}Y$$

Σ' has swapping: $\mathbf{swap} : \delta\Sigma(X) \cong \Sigma'(X, \delta(X))$

Σ' is strong: $\mathbf{str} : \Sigma(X, Y)\hat{\otimes}Z \to \Sigma'(X, Y\hat{\otimes}Z)$

Linear Substitution Algebra

A linear substitution algebra is a triple (X, σ, ν) where $X \in \mathcal{B}$, $\sigma : \delta(X) \hat{\otimes} X \to X$ and $\nu : I \to \delta(X)$ with

$$
\begin{array}{ccc}
I \otimes X & \xrightarrow{\cong} & X \\
{\scriptstyle \nu \hat{\otimes} \mathrm{id}} \downarrow & \nearrow {\scriptstyle \sigma} & \\
\delta(X) \hat{\otimes} X & &
\end{array}
$$

$$
\begin{array}{ccccccc}
\delta\Sigma_{\mathrm{s}}(X) \hat{\otimes} X & \xrightarrow{\cong} & \Sigma'_{\mathrm{s}}(X, \delta(X)) \hat{\otimes} X & \xrightarrow{\mathbf{str}} & \Sigma'_{\mathrm{s}}(X, \delta(X) \hat{\otimes} X) & \xrightarrow{\Sigma'_{\mathrm{s}}(\mathrm{id}, \sigma)} & \Sigma'_{\mathrm{s}}(X, X) \\
{\scriptstyle \delta(\sigma) \otimes \mathrm{id}} \downarrow & & & & & & \downarrow {\scriptstyle \sigma + \sigma} \\
\delta(X) \otimes X & & & \xrightarrow{\sigma} & & & X
\end{array}
$$

Linear Substitution Algebra

A linear substitution algebra is a triple (X, σ, ν) where $X \in \mathcal{B}$, $\sigma : \delta(X)\hat{\otimes}X \to X$ and $\nu : I \to \delta(X)$ with

$$\begin{array}{ccc} I \otimes X & \xrightarrow{\cong} & X \\ {\scriptstyle \nu\hat{\otimes}\mathrm{id}}\downarrow & \nearrow_{\sigma} & \\ \delta(X)\hat{\otimes}X & & \end{array}$$

$$\begin{array}{ccccccc} \delta\Sigma_{\mathrm{s}}(X)\hat{\otimes}X & \xrightarrow{\cong} & \Sigma'_{\mathrm{s}}(X, \delta(X))\hat{\otimes}X & \xrightarrow{\mathbf{str}} & \Sigma'_{\mathrm{s}}(X, \delta(X)\hat{\otimes}X) & \xrightarrow{\Sigma'_{\mathrm{s}}(\mathrm{id},\sigma)} & \Sigma'_{\mathrm{s}}(X, X) \\ {\scriptstyle \delta(\sigma)\otimes\mathrm{id}}\downarrow & & & & & & \downarrow{\scriptstyle \sigma+\sigma} \\ \delta(X) \otimes X & & & \xrightarrow{\sigma} & & & X \end{array}$$

LinSubstAlg $\cong$ SymOp

Linear Substitution Algebra

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$\varphi : \Sigma(TV) \to TV$ has induced linear substitution algebra