Weak equivalences between algebraic weak ω-categories

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[3] JSPS KAKENHI Grant Number JP21K20329, JP23K12960 & P24KJ0126

Weak ω-categories (Batanin, Leinster)

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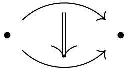
$\bullet$	$\bullet \longrightarrow \bullet$		$\cdots$

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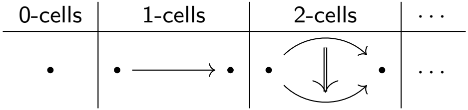

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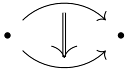
$\bullet$	$\bullet \longrightarrow \bullet$		$\cdots$

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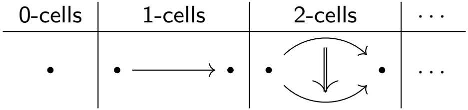

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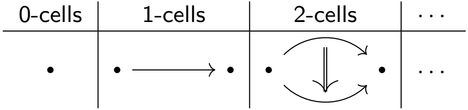

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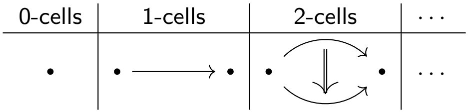

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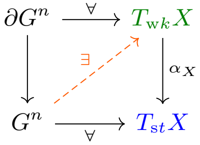

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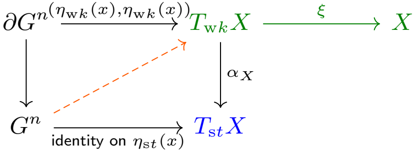

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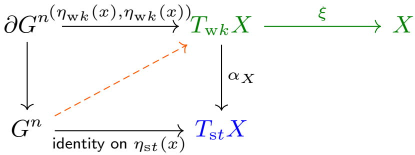

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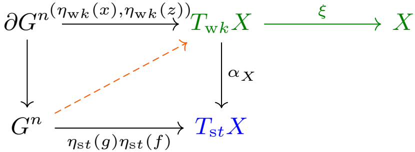

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A weak ω-category has "all" the operations that a strict ω-category has, including what one usually think of as relations.

There are operations that take a 1-cell $f\colon x \to y$ as input and spit out:

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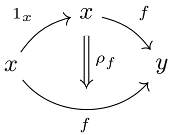

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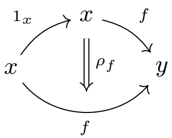

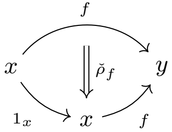

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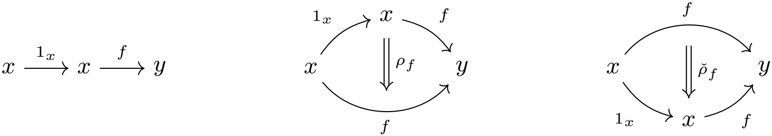

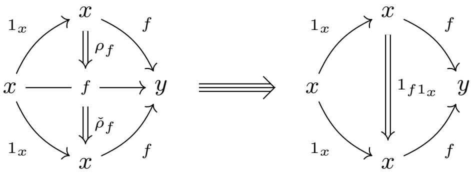

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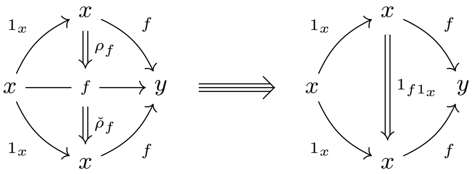

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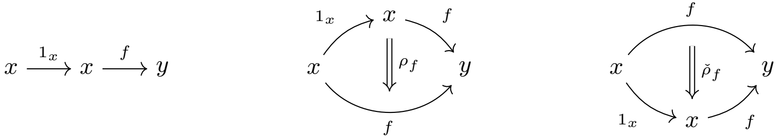

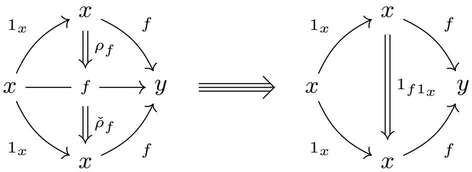

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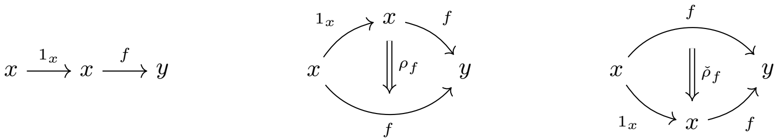

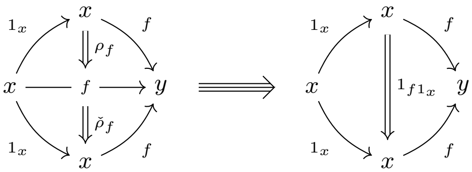

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The fun/tricky part is correctly identifying *what operations one needs* in a given situation.

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Theorem (Fujii-Hoshino-M.)

The class of weak equivalences enjoys the 2-out-of-3 property.
That is, if any two of F, G and GF are weak equivalences then so is the third.

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The latter is still non-trivial for weak ω-categories!

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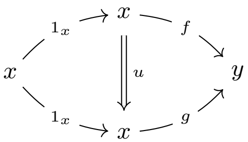

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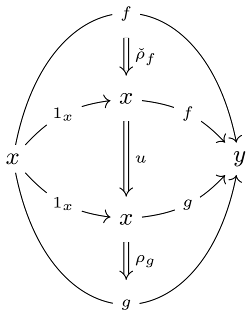

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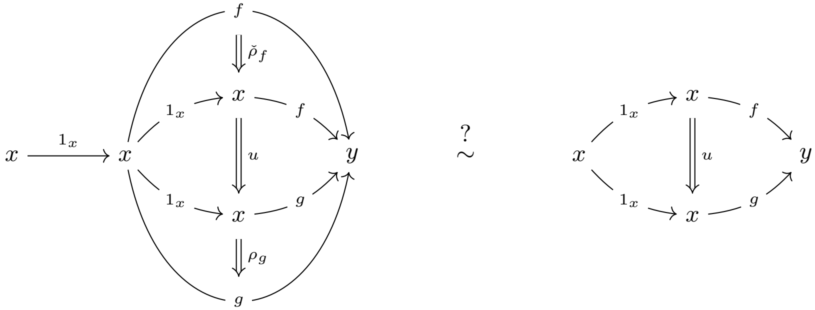

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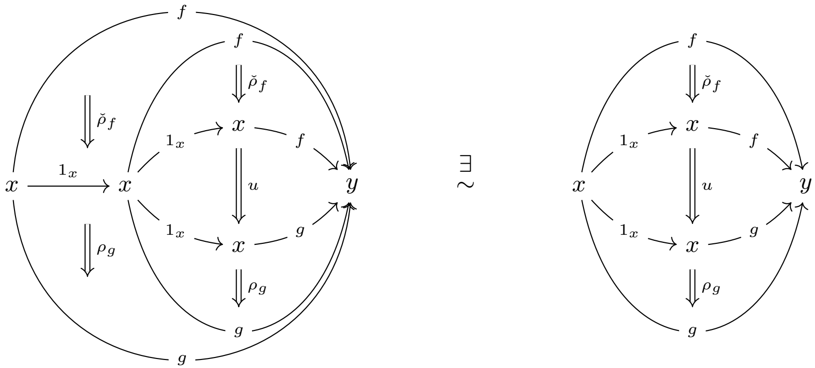

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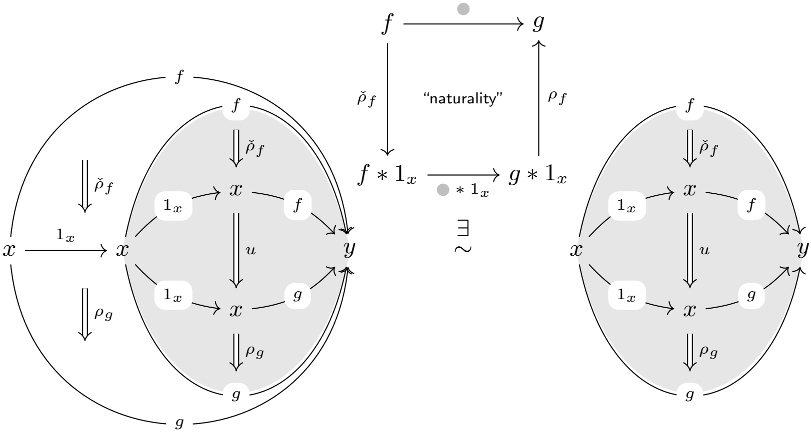

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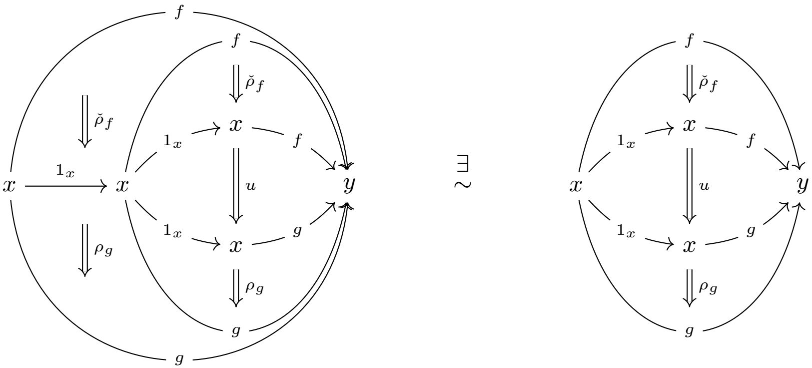

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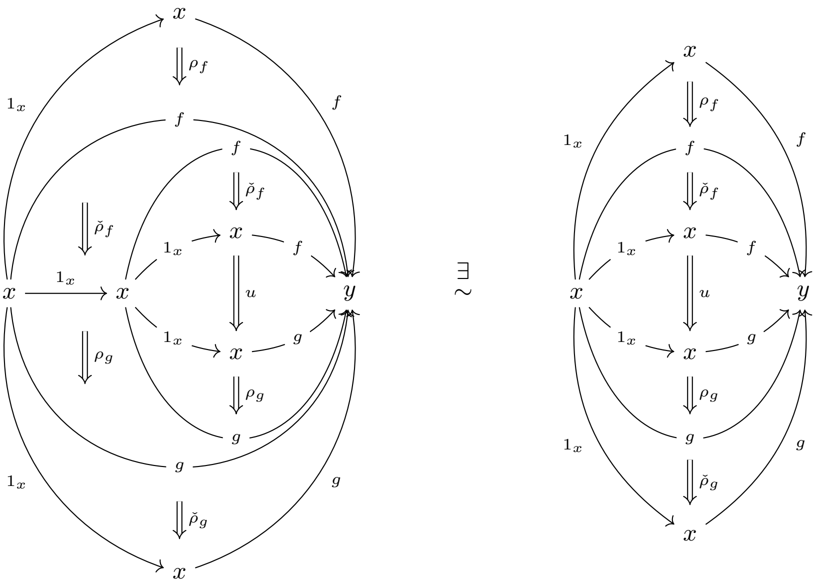

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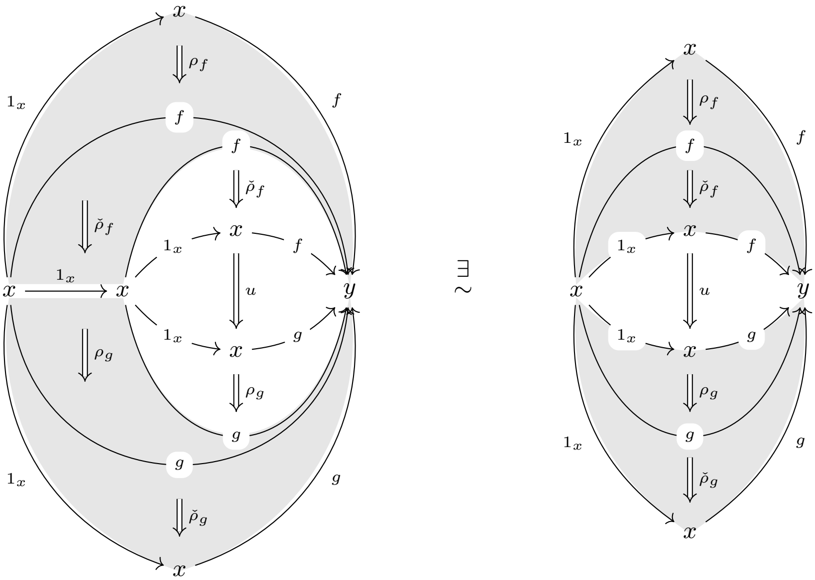

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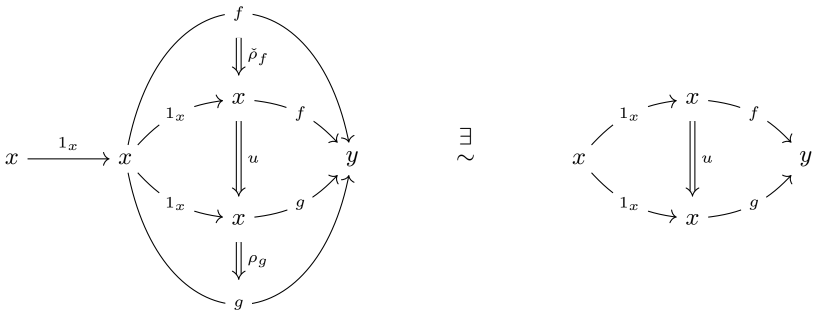

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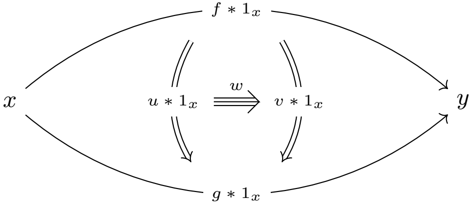

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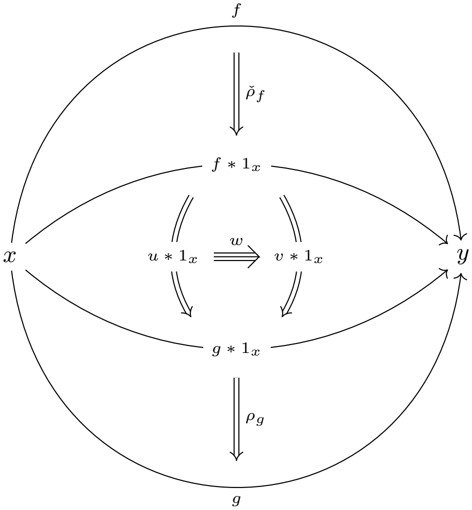

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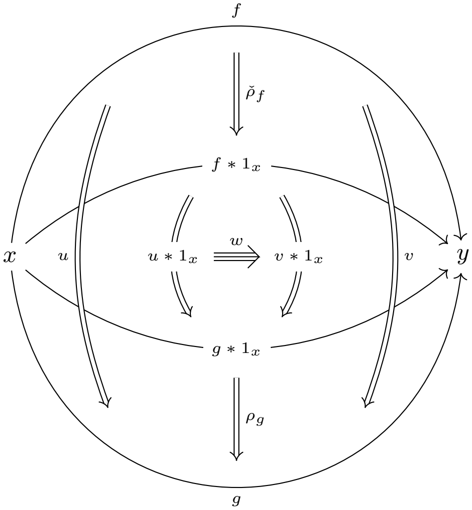

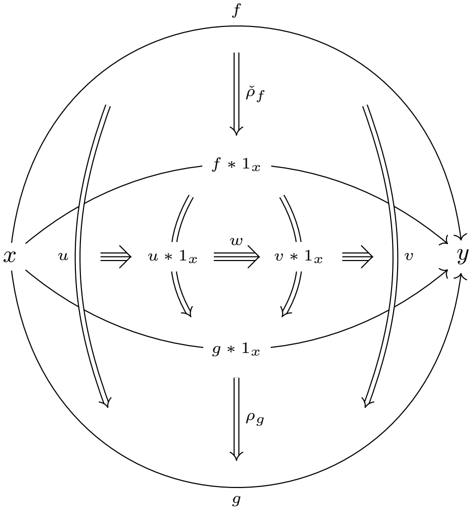
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