Semicartesian categories of relations

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- Categorical generalizations of **Rel**;
- Categorical axiomatizations of various (dagger) categories.

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- Prime example: Connes' noncommutative standard model.

Noncommutative dictionary

Mathematical structure	Noncommutative generalization
Locally compact Hausdorff spaces	C*-algebras
Compact Hausdorff spaces	Unital C*-algebras
Connected component	Projections
Measure spaces	Von Neumann algebras
Riemannian manifolds	Spectral triples
Compact groups	Compact matrix quantum groups
Banach spaces	Operator spaces
Graphs	Operator systems
Sets	Sums of matrix algebras

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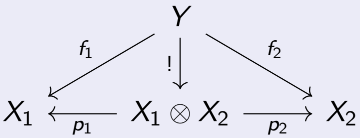
- **qSet** $:=$ $\mathbf{WStar}_{\mathrm{HA}}^{\mathrm{op}}$ is noncommutative generalization of **Set**.

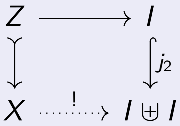
Quantum (Grothendieck) topoi?

Theorem (Kornell)

The category **qSet**

1. *is complete and cocomplete,*
2. *is semicartesian closed;*
3. *has, for each pair of morphisms $f_1 \colon Y \to X_1$ and $f_2 \colon Y \to X_2$, <u>at most</u> one morphism making the left diagram below commute,*
4. *and has, for every monic $Z \rightarrowtail X$, a unique "classical" morphism from X to the coproduct $I \uplus I$ making the right diagram below into a pullback square:*





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- Subobjects of $A \otimes B$ don't yield a relevant calculus of relations.

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In a dagger quantaloid $\mathbf{R}$, a morphism $f : X \to Y$ is called a map if $f^\dagger \circ f \geq \mathrm{id}_X$ and $f \circ f^\dagger \leq \mathrm{id}_Y$.

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A quantum relation on a quantum set is called a preorder if it is reflexive, and transitive.

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Example (Kornell, L., Mislove)

The category **qPreOrd** of preordered quantum sets is complete, cocomplete, symmetric monoidal closed, and **PreOrd**-enriched.

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- Fuzzification = internalization in V-**Rel** ?
- Dagger compact quantaloids form a unifying setting.

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Definition

A quantum suplattice is an $\mathcal{D}$-algebra.

- Several theorems (existence of Galois connections, Knaster-Tarski Fixpoint Theorem) carry over to quantum suplattices;
- Proofs entirely based on the categorical structure of **qRel**.

Existence of monads

Theorem

Given:

- *A symmetric monoidal closed category* $\mathbf{S}$ *with internal hom* $[-,-]$*;*
- *A compact closed category* $\mathbf{R}$*;*
- *A strict monoidal functor* $J : \mathbf{S} \to \mathbf{R}$ *that is bijective on objects;*
- *An object* $\Omega \in \mathbf{S}$ *and a morphism* $c : J\Omega \to I$ *such that* $\mathbf{S}(A, \Omega) \to \mathbf{R}(JA, I)$, $f \mapsto c \circ Jf$ *is a bijection for each* $A \in \mathbf{S}$.

Then J *has a right adjoint whose action on objects is given by* $X \mapsto [J^{-1}(X^*), \Omega]$.

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- power set monad: $\mathbf{S} = \mathbf{Set}$, $\mathbf{R} = \mathbf{Rel}$;
- quantum power set monad: $\mathbf{S} = \mathbf{qSet}$, $\mathbf{R} = \mathbf{qRel}$
- lower set monad: $\mathbf{S} = \mathbf{PreOrd}$, $\mathbf{R} = \mathbf{MonRel}$;
- quantum lower set monad: $\mathbf{S} = \mathbf{qPreOrd}$, $\mathbf{R} = \mathbf{qMonRel}$.

Monotone relations

Definition

A monotone relation $r : (X, \sqsubseteq_X) \to (Y, \sqsubseteq_Y)$ between preordered sets is a relation $r : X \to Y$ such that $\sqsupseteq_Y \circ r = r = r \circ \sqsupseteq_X$.

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Corollary

The category **qMonRel** := **MonRel**(**qRel**) *is compact closed.*

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Theorem

- *(1)-(2)* $\implies$ $\mathbf{R}$ *is a quantaloid;*
- *(1)-(4)* $\implies$ *homsets of* $\mathbf{R}$ *are complete orthomodular lattices;*
- *(1)-(4)* $\implies$ $\mathrm{Maps}(\mathbf{R})$ *is semicartesian.*

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- (b) follows if $p \in \mathbf{Proj}(X)$, $\ker(p) = 0 \implies p \geq \mathrm{id}_X$.
- Power objects $\implies$ monoidal closure of $\mathrm{Maps}(\mathbf{R})$???

Conclusions

- Quantization by internalization in **qRel**;
- Preliminary axioms for ‘semicartisian categories of relations’ based on dagger structures;
- Examples: **Rel** and **qRel**;
- Axioms imply existence quantaloid structure, orthomodular structure of homsets;
- Monoidal closure $\mathrm{Maps}(\mathbf{R})$ implies power objects;
- Existence and theorems on quantum suplattices follow from abstract principles.

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