2-stacks over bisites

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Sieves

Let $\mathcal{C}$ be a category with pullbacks.

Definition.

A **sieve** S on an object $C \in \mathcal{C}$ is a collection of morphisms with codomain C that is closed under precomposition with morphisms of $\mathcal{C}$.

The sieve S can also be seen as a subfunctor of $y(C)$, i.e. a natural transformation

$$S\colon R \Rightarrow y(C)$$

with injective components.

Grothendieck topology on a category

Definition.

A **Grothendieck topology** τ on $\mathcal{C}$ is an assignment for each object $C \in \mathcal{C}$ of a collection $\tau(C)$ of sieves on C, called **covering sieves**, in a way such that

(T0) the maximal sieve $y(C)$ is in $\tau(C)$;

(T1) if $S \in \tau(C)$, then for every arrow $f\colon D \to C$ we have that $f^*S \in \tau(D)$;

(T2) if $S \in \tau(C)$ and R is a sieve on C such that for every $f\colon D \to C$ in S we have that $f^*R \in \tau(D)$, then $R \in \tau(C)$.

Descent data

Definition.

Let $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}at$ be a pseudofunctor and let S be a sieve on $C \in \mathcal{C}$. A **descent datum on S for F** is an assignment for every morphism $D \xrightarrow{f} C$ in S of an object $W_f \in F(D)$ and, for every pair of composable morphisms $E \xrightarrow{g} D \xrightarrow{f} C$ with $f \in S$, of an isomorphism $\varphi^{f,g}\colon g^* W_f \xrightarrow{\simeq} W_{f\circ g}$ such that, given morphisms $L \xrightarrow{h} E \xrightarrow{g} D \xrightarrow{f} C$ with $f \in S$, the following diagram is commutative

$$\begin{array}{ccc}
h^*(g^* W_f) & \xrightarrow{h^*\varphi^{f,g}} & h^*(W_{f\circ g}) \\
\downarrow \wr & & \downarrow \varphi^{f\circ g, h} \\
(g\circ h)^*(W_f) & \xrightarrow[\varphi^{f,g\circ h}]{} & W_{f\circ g\circ h}.
\end{array}$$

Descent data

Definition.

This descent datum is called **effective** if there exist an object $W \in F(C)$ and, for every morphism $D \xrightarrow{f} C \in S$, an isomorphism

$$\psi^f : f^*(W) \xrightarrow{\simeq} W_f$$

such that, given morphisms $E \xrightarrow{g} D \xrightarrow{f} C$ with $f \in S$, the following diagram is commutative

$$\begin{array}{ccc} g^*(f^*(W)) & \xrightarrow{g^*\psi^f} & g^*(W_f) \\ \downarrow \wr & & \downarrow \varphi^{f,g} \\ (f \circ g)^* W & \xrightarrow{\psi^{f \circ g}} & W_{f \circ g}. \end{array}$$

Definition of stack

Definition.

A pseudofunctor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}at$ is a **stack** if it satisfies the following conditions:

- Every descent datum on a covering sieve S for F is effective;
- Given a covering sieve S on C, objects X and Y of $F(C)$ and for every $f\colon D \to C$ in S a morphism $w_f\colon f^*X \to f^*Y$ in $F(D)$ such that $g^*(w_f) = w_{f\circ g}$, there exists a unique morphism $w\colon X \to Y$ such that $f^*w = w_f$.

A useful characterization

Proposition (Street).

Let $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}at$ *be a pseudofunctor. The following facts are equivalent:*

(1) F *is a stack;*

(2) for every object $C \in \mathcal{C}$ *and every covering sieve* $S\colon R \Rightarrow y(C)$ *in* $\tau(C)$ *the functor*

$$- \circ S\colon [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at](y(C), F) \longrightarrow [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at](R, F)$$

is an equivalence of categories.

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is an equivalence of categories.

- essentially surjective $\approx$ every descent datum is effective
- fully-faithful $\approx$ every matching family of morphisms has a unique amalgamation

A useful characterization

$$F(C) \xrightarrow{\Gamma} [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at](y(C), F) \xrightarrow{-\circ S} [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at](R, F)$$

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- $\alpha\colon R \Rightarrow F$ pseudonatural transformation is given by:
  - $D \in \mathcal{C}$, $\alpha_D\colon R(D) \to F(D)$ functor that sends $D \xrightarrow{f} C$ to $W_f \in F(D)$
  - $E \xrightarrow{g} D$, $\alpha_g\colon \alpha_E \circ R(g) \Rightarrow F(g) \circ \alpha_D$ pseudonatural transformation of components $\varphi^{f,g}\colon W_{f\circ g} \xrightarrow{\simeq} g^* W_f$

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$$\alpha \cong (- \circ S) \circ \Gamma(W) \qquad \approx \qquad \psi \colon \Gamma(W) \circ S \Rrightarrow \alpha, \;\; \psi^f \colon f^* W \xrightarrow{\simeq} W_f$$

A useful characterization

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- $X, Y \in F(C)$, $m\colon ((- \circ S) \circ \Gamma)(X) \Rrightarrow ((- \circ S) \circ \Gamma)(Y)$ modification is given by:
  - $D \in \mathcal{K}$, $m_D\colon (\Gamma(X))_D \Rightarrow (\Gamma(Y))_D$ natural transformation of components $w_f\colon f^* X \to f^* Y$

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$$F(C) \xrightarrow{\Gamma} [\mathcal{C}^{op}, \mathcal{C}at](y(C), F) \xrightarrow{-\circ S} [\mathcal{C}^{op}, \mathcal{C}at](R, F)$$

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$m = (- \circ S) \circ \Gamma(X \xrightarrow{w} Y) \qquad \approx \qquad f^* w = w_f \quad$ for every $f \in S$

Bisieves

Let $\mathcal{K}$ be a small 2-category with bi-iso-comma objects.

Definition (Street).

A **bisieve** S over $C \in \mathcal{K}$ is a fully faithful arrow $S\colon R \Rightarrow y(C)$ in $\mathcal{Bicat}(\mathcal{K}^{\mathrm{op}}, \mathcal{Cat})$.

- for our purposes we can consider S injective on objects;
- S is closed under precomposition only up to isomorphism;
- all 2-cells are in S.

Grothendieck topology on a 2-category

Definition (Street).

A **bitopology** τ on $\mathcal{K}$ is an assignment for each object $\mathcal{C} \in \mathcal{K}$ of a collection $\tau(C)$ of bisieves on C, called **covering bisieves**, in a way such that

(T0) the identity of $y(C)$ is in $\tau(C)$;

(T1) for all $S\colon R \to y(C)$ in $\tau(\mathcal{Y})$ and all arrows $f\colon D \to C$ in $\mathcal{K}$, the bi-iso-comma object

$$\begin{array}{ccc} P & \longrightarrow & y(D) \\ \big\downarrow & \overset{\cong}{\swarrow} & \big\downarrow{\scriptstyle -\circ f} \\ R & \xrightarrow[S]{} & y(C) \end{array}$$

has the top arrow is in $\tau(D)$;

(T2) being a covering bisieve can be checked locally.

2-stacks

Definition (C.).

Let $(\mathcal{K}, \tau)$ be a bisite. A trihomomorphism $F : \mathcal{K}^{\mathrm{op}} \to \mathcal{B}icat$ is a **2-stack** if for every object $C \in \mathcal{K}$ and every bisieve $S\colon R \Rightarrow y(C)$ in $\tau(C)$ the pseudofunctor

$$- \circ S\colon \mathcal{T}ricat(\mathcal{K}^{\mathrm{op}}, \mathcal{B}icat)(y(C), F) \longrightarrow \mathcal{T}ricat(\mathcal{K}^{\mathrm{op}}, \mathcal{B}icat)(R, F)$$

is a biequivalence.

Towards explicit conditions

Theorem (Buhné).

Let $F\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{B}icat$ *be a trihomomorphism. For every* $C \in \mathcal{K}$ *there exists a biequivalence*

$$\Gamma\colon F(C) \longrightarrow \mathcal{T}ricat(\mathcal{K}^{\mathrm{op}}, \mathcal{B}icat)(y(C), F)$$

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Remark.

Assuming the axiom of choice, biequivalence means:

(1) surjective on equivalence classes of objects;

(2) essentially surjective on morphisms;

(3) fully-faithful on 2-cells.

Surjectivity on equivalence classes of objects

$$F(C) \xrightarrow{\Gamma} \mathcal{Tricat}(\mathcal{K}^{\mathrm{op}}, \mathcal{Bicat})(y(C), F) \xrightarrow{-\circ S} \mathcal{Tricat}(\mathcal{K}^{\mathrm{op}}, \mathcal{Bicat})(R, F)$$

Surjectivity on equivalence classes of objects

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$\alpha\colon R \Rightarrow F$ tritransformation is given by:

- $D \in \mathcal{K}$, $\alpha_D\colon R(D) \to F(D)$ pseudofunctor that sends $D \xrightarrow{f} C$ to $W_f \in F(D)$ and $\eta\colon f \Rightarrow f'$ to $W_f \xrightarrow{W_\eta} W_{f'}$
- $E \xrightarrow{g} D$, $\alpha_g\colon \alpha_E \circ R(g) \Rightarrow F(g) \circ \alpha_D$ pseudonatural transformation of components $\varphi^{f,g}\colon W_{\widetilde{f\circ g}} \xrightarrow{\sim} g^* W_f$
- $L \xrightarrow{h} E \xrightarrow{g} D$, $\beta^{g,h}$ invertible modification of components

$$\begin{array}{ccccc}
W_{\widetilde{\widetilde{f\circ g}\circ h}} & \xrightarrow{\varphi^{\widetilde{f\circ g},h}} & g^* W_{\widetilde{f\circ g}} & \xrightarrow{h^*\varphi^{f,g}} & h^* g^* W_f \\
\Downarrow\wr & & \overset{\sim}{\Leftarrow}\ \beta_f^{g,h} & & \downarrow\wr \\
W_{\widetilde{f\circ g\circ h}} & & \xrightarrow[\varphi^{f,g\circ h}]{} & & (g\circ h)^* W_f
\end{array}$$

Surjectivity on equivalence classes of objects

$$F(C) \xrightarrow{\Gamma} \mathcal{T}ricat(\mathcal{K}^{\mathrm{op}}, \mathcal{B}icat)(y(C), F) \xrightarrow{-\circ S} \mathcal{T}ricat(\mathcal{K}^{\mathrm{op}}, \mathcal{B}icat)(R, F)$$

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W_{\widetilde{f\circ g\circ h}} & & \xrightarrow[\varphi^{f,g\circ h}]{} & & (g\circ h)^* W_f
\end{array}$$

$$\alpha \simeq ((- \circ S) \circ \Gamma)(W) \qquad \approx \qquad \psi \colon \alpha \Rrightarrow \Gamma(W) \circ S,\ \psi^f \colon W_f \xrightarrow{\simeq} f^* W$$

Essential surjectivity on morphisms

$$F(C) \xrightarrow{\Gamma} \mathcal{Tricat}(\mathcal{K}^{\mathrm{op}}, \mathcal{Bicat})(y(C), F) \xrightarrow{-\circ S} \mathcal{Tricat}(\mathcal{K}^{\mathrm{op}}, \mathcal{Bicat})(R, F)$$

$X, Y \in F(C)$, $m\colon ((- \circ S) \circ \Gamma)(X) \Rrightarrow ((- \circ S) \circ \Gamma)(Y)$ trimodification is given by:

- $D \in \mathcal{K}$, $m_D\colon (\Gamma(X))_D \Rightarrow (\Gamma(Y))_D$ pseudonatural transformation of components $w_f\colon f^*X \to f^*Y$ (the pseudonaturality gives isomorphic 2-cells $w_\eta\colon F(\eta)_Y \circ w_f \Rightarrow w_{f'} \circ F(\eta)_X$)
- $E \xrightarrow{g} D$, m_g invertible modification of components $\varphi^{f,g}$ relating $g^* w_f$ and $w_{\widetilde{f \circ g}}$

The axioms of trimodification yield the cocycle condition on morphisms.

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The axioms of trimodification yield the cocycle condition on morphisms.

$$m \cong ((-\circ S)\circ\Gamma)(X \xrightarrow{w} Y) \qquad \approx \qquad \psi\colon m \Rrightarrow \Gamma(w)\circ S, \psi^f\colon w_f \overset{\simeq}{\Rightarrow} f^*w$$

Fully-faithfulness on 2-cells

$$F(C) \xrightarrow{\Gamma} \mathcal{T}ricat(\mathcal{K}^{\mathrm{op}}, \mathcal{B}icat)(y(C), F) \xrightarrow{-\circ S} \mathcal{T}ricat(\mathcal{K}^{\mathrm{op}}, \mathcal{B}icat)(R, F)$$

$a, b \colon X \to Y$, $p \colon ((-\circ S)\circ\Gamma)(a) \Rrightarrow ((-\circ S)\circ\Gamma)(b)$ perturbation is given by:

- $d \in \mathcal{K}$, $p_D \colon (\Gamma(a))_D \Rrightarrow (\Gamma(b))_D$ modification of components $w_f \colon f^*a \Rightarrow f^*b$

The axiom of perturbation yields the compatibility with respect to composition.

Fully-faithfulness on 2-cells

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The axiom of perturbation yields the compatibility with respect to composition.

$$p = (- \circ S) \circ \Gamma(a \overset{w}{\Rightarrow} b) \qquad \approx \qquad w_f = f^*w \quad \text{for every } f \in S$$

2-stacks: explicit characterization

Theorem (C.).

A trihomomorphism $F\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{B}icat$ *is a 2-stack if and only if for every* $C \in \mathcal{K}$ *and every bisieve* $S \in \tau(C)$ *the following conditions are satisfied:*

(O) every weak descent datum for S *of elements of* F *is weakly effective;*

(M) every descent datum for S *of morphisms of* F *is effective;*

(2C) every matching family for S *of 2-cells of* F *has a unique amalgamation.*

Principal 2-bundles

Let $\mathcal{K}$ be a small $(2,1)$-category with iso comma objects.

Definition (C.).

Let $\mathcal{X}$ and $\mathcal{Y}$ be objects of $\mathcal{K}$ with fixed actions of the internal 2-group $\mathcal{G}$ on them. We say that the morphism $g\colon \mathcal{Y} \to \mathcal{X}$ is **2-locally trivial** if there exists a covering bisieve $S\colon R \Rightarrow \mathcal{K}(-,\mathcal{X})$ in $\tau(\mathcal{X})$ such that, for every $f\colon \mathcal{U} \to \mathcal{X}$ in the bisieve S, the iso-comma object of g along f is equivalent to $\mathcal{G} \times \mathcal{U}$ over $\mathcal{U}$ via a $\mathcal{G}$-equivariant equivalence.

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Definition (C.).

Let $\mathcal{Y}$ be an object of $\mathcal{K}$. A **principal $\mathcal{G}$-2-bundle over $\mathcal{Y}$** is an object $\mathcal{P} \in \mathcal{K}$ equipped with an action $p\colon \mathcal{G} \times \mathcal{P} \to \mathcal{P}$ and a $\mathcal{G}$-equivariant 2-locally trivial morphism $\pi_{\mathcal{P}}\colon \mathcal{P} \to \mathcal{Y}$.

Quotient pre-2-stacks

Definition (C.).

The **quotient pre-2-stack** $[\mathcal{X}/\mathcal{G}]\colon \mathscr{K}^{\mathrm{op}} \to 2\mathcal{C}at$ is defined as follows:

- for every object $\mathcal{Y} \in \mathscr{K}$ we define $[\mathcal{X}/\mathcal{G}](\mathcal{Y})$ as the 2-category of pairs $(\mathcal{P}, \alpha)$ where $\pi_{\mathcal{P}}\colon \mathcal{P} \to \mathcal{Y}$ is a principal $\mathcal{G}$-2-bundle over $\mathcal{Y}$ and $\alpha\colon \mathcal{P} \to \mathcal{X}$ is a $\mathcal{G}$-equivariant morphism;
- for every morphism $f\colon \mathcal{Z} \to \mathcal{Y}$ in $\mathscr{K}$, we define the 2-functor
$$[\mathcal{X}/\mathcal{G}](f) = f^*\colon [\mathcal{X}/\mathcal{G}](\mathcal{Y}) \to [\mathcal{X}/\mathcal{G}](\mathcal{Z})$$
via iso-comma object along f;
- for every 2-cell $\Lambda\colon f \Rightarrow g\colon \mathcal{Z} \to \mathcal{Y}$, we define $[\mathcal{X}/\mathcal{G}](\Lambda)$ using the universal property of the iso-comma object.

$[\mathcal{X}/\mathcal{G}]$ is a 2-stack

Theorem (C.).

Let $\mathcal{K}$ be a bicocomplete and finitely bicomplete $(2,1)$- category such that comma objects preserve bicolimits in $\mathcal{K}$ and let τ be a subcanonical bitopology on it. Then $[\mathcal{X}/\mathcal{G}]$ is a 2-stack.

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Let $\mathcal{K}$ be a bicocomplete and finitely bicomplete $(2,1)$- category such that comma objects preserve bicolimits in $\mathcal{K}$ and let τ be a subcanonical bitopology on it. Then $[\mathcal{X}/\mathcal{G}]$ is a 2-stack.

Key idea of the proof:

Proposition (C.).

Let τ be a subcanonical topology on $\mathcal{K}$ and let $S\colon R \Rightarrow y(\mathcal{C})$ be a covering bisieve. Then $\mathcal{C} = \sigma\text{-bicolim}\,\Pi$, where $\Pi\colon \int R \to K$ is the 2-functor of projection on the first component.

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