Two-dimensional limit theories (enhanced)

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- But problems emerge: need to consider 2-cats with special maps (extra structure)

→ enhanced 2-d lim. sketches

Internal categories

- Internal category is a diagram

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\begin{tikzcd}
A_1 \times_{A_0} A_1 \arrow[r, shift left=4, "\pi_1"] \arrow[r, "m"] \arrow[r, shift right=4, "\pi_2"'] & A_1 \arrow[r, shift left=4, "s"] \arrow[r, shift right=4, "t"'] & A_0 \arrow[l, "i"']
\end{tikzcd}
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such that $si = ti = 1$ & satisfying associativity & unit laws.

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- **Internal functor** $F : A \longrightarrow B$:

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B_1 \times_{B_0} B_1 \arrow[r, shift left=2, "\pi_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "\pi_2"'] & B_1 \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & B_0 \arrow[l, "i" description]
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Internal cats in Cat - double categories

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v. arrows, squares — objects, h. arrows

$$\boxed{\begin{tikzcd} a \arrow[r] \arrow[d] \arrow[dr, phantom, "\varphi"] & b \arrow[d] \\ c \arrow[r] & d \end{tikzcd}} \quad \begin{array}{c} \xmapsto{s} \\ \xmapsto{t} \end{array} \quad \begin{array}{c} \boxed{a \longrightarrow b} \\ \boxed{c \longrightarrow d} \end{array}$$

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but

$$\begin{tikzcd} A_1 \times_{A_0} A_1 \arrow[r, "m"] \arrow[d, "F_1 \times_{F_0} F_1"'] \arrow[dr, phantom, "\Swarrow"] & A_1 \arrow[d, "F_1"] \arrow[dr, phantom, "\Searrow"] & A_0 \arrow[l, "i"'] \arrow[d, "F_0"] \\ B_1 \times_{B_0} B_1 \arrow[r, "m"'] & B_1 & B_0 \arrow[l, "i"] \end{tikzcd}$$

only composition & unit laxly.

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only composition & unit laxly.

- Lax natural transformation but strictly natural at s, t (also π_1 π_2).
- Must treat s, t, π_1, π_2 as special :

$$\begin{tikzcd} A_1 \times_{A_0} A_1 \arrow[r, shift left=2, "\pi_1"] \arrow[r, squiggly, "m" description] \arrow[r, shift right=2, "\pi_2"'] & A_1 \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & A_0 \arrow[l, squiggly, "i" description] \end{tikzcd}$$

$\longrightarrow$ for special, $\rightsquigarrow$ for general

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- Or a pseudomonoid

strict

$$
\begin{tikzcd}
A^2 \arrow[r, shift left=2, "\pi_1"] \arrow[r, squiggly, "m" description] \arrow[r, shift right=2, "\pi_2"'] & A & 1 \arrow[l, squiggly, "i"']
\end{tikzcd}
$$

of pseudo-double cats
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Summary so Far

Need to move from diags like

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- → enhanced 2-cat theory & enhanced 2-d limit sketches

Enhanced 2-categories

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tight maps   loose maps

- Many examples :-

$MonCat_{s,\ell}$: monoidal cats, strict & lax mon. Functors

- or any combination.

As $\mathcal{F}$-enriched cats

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- $\mathcal{F}$-functor $F : \mathbb{A} \to \mathbb{B}$ is a 2-Functor $F : A_\lambda \to B_\lambda$ preserving tightness

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- $\mathcal{F}$-natural transformation = 2-nat transf. with tight components.

$\mathcal{F}$-weighted limits

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  i.e. <u>lims of diagrams of tight morphisms univ. prop wrt loose morphisms too</u>.

- E.g. Tight pullbacks

$$
\begin{tikzcd}
 & & A \arrow[dr, "f"] & \\
D \arrow[rru, squiggly, bend left, "u"] \arrow[r, squiggly, "\exists !"] \arrow[rrd, squiggly, bend right, "v"'] & P \arrow[ur, "p"] \arrow[dr, "q"'] \arrow[rr, phantom, "\lrcorner"] & & C \\
 & & B \arrow[ur, "g"'] &
\end{tikzcd}
$$

Enriched limit sketches

Ehresmann 1968

- A limit sketch is a category $\mathcal{T}$ with a collection of

$$(D : \mathcal{J} \to \mathcal{C},\ p : \Delta X \to D)$$

diagrams　　　　cones over diagrams

Enriched limit sketches

- A $\mathcal{V}$-limit sketch (Kelly 1982) is a $\mathcal{V}$-cat $\mathcal{T}$ with a collection of
$$\big(W : \mathcal{J} \to \mathcal{V},\ D : \mathcal{J} \to \mathcal{C},\ p : W \to \mathcal{C}(X, D-)\big)$$
weights, diagrams, weighted cones

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$$\left(\underset{\text{weights}}{W : \mathcal{J} \to \mathcal{V}},\ \underset{\text{diagrams}}{D : \mathcal{J} \to \mathcal{C}},\ \underset{\text{weighted cones}}{p : W \to \mathcal{C}(x, D-)} \right)$$

- $\mathcal{V} = \mathrm{Cat} \rightarrow$ Two-dim. lim sketch

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weights — diagrams — weighted cones

- $\mathcal{V} = \mathbf{Cat}$ → Two-dim. lim sketch
- $\mathcal{V} = \mathcal{F}$ → enhanced 2-d lim sketch

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weights — diagrams — weighted cones

- $\mathcal{V} = \mathrm{Cat} \to$ Two-dim. lim sketch
- $\mathcal{V} = \mathcal{F} \to$ enhanced 2-d lim sketch
- Model of T in $\mathcal{V}$-sketch $\mathcal{C}$ is a sketch morphism: $\mathcal{V}$-functor preserving weighted cones.

Models of $\mathcal{F}$-sketches

$\mathrm{Mod}_\ell(T, \mathcal{C})$ is $\mathcal{F}$-cat of models & lax maps.

lax

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lax

- loose morph $M \overset{f}{\rightsquigarrow} N$ is lax natural transf

$$\begin{tikzcd} T_\lambda \arrow[r, bend left=50, "M"] \arrow[r, bend right=50, "N"'] \arrow[r, phantom, "f\Downarrow"] & \mathcal{C}_\lambda \end{tikzcd}$$

such that

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is 2-natural:

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is an identity 2-cell for $\alpha : X \to Y$ tight.

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- tight = $\mathcal{F}$-nat transformation

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$$\begin{tikzcd} T_\lambda \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "f \Downarrow"] & \mathcal{C}_\lambda \end{tikzcd}$$
such that
$$\begin{tikzcd} T_\tau \arrow[r, "J"] & T_\lambda \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "f \Downarrow"] & \mathcal{C}_\lambda \end{tikzcd}$$
is 2-natural:

ie.
$$\begin{tikzcd} MX \arrow[r, squiggly, "f_X"] \arrow[d, "M\alpha"'] \arrow[dr, phantom, "f_\alpha \Downarrow"] & NX \arrow[d, "N\alpha"] \\ MY \arrow[r, squiggly, "f_Y"'] & NY \end{tikzcd}$$
is an identity 2-cell for $\alpha : X \to Y$ tight.

- tight = $\mathcal{F}$-nat Transformation

- 2-cells are modifications.

T-cats of models ctd

- $\omega \in \{s, p, l, c\}$

strict pseudo lax colax

$\mathcal{F}$-cats of models ctd

- $\omega \in \{s, p, l, c\}$

strict, pseudo, lax, colax

- F-cat of models $\mathrm{Mod}_\omega(T, \mathcal{C})$ involves ω-natural transformations

$\mathbb{F}$-cats of models ctd

- $\omega \in \{ s, p, l, c \}$

strict pseudo lax colax

- $\mathbb{F}$-cat of models $\mathrm{Mod}_\omega(T, \mathcal{C})$ involves <u>ω-natural transformations</u>

- Each in fact an <u>$\mathbb{F}$-sketch</u>: take <u>levelwise weighted cones</u>.

Examples of $\mathcal{F}$-sketches

① C-sketch for pseudocats in an $\mathcal{F}$-cat:

$$\begin{tikzcd}
A_1 \times_{A_0} A_1 \arrow[r, shift left=3, "\pi_1"] \arrow[r, squiggly, "m" description] \arrow[r, shift right=3, "\pi_2"'] & A_1 \arrow[r, shift left=3, "s"] \arrow[r, shift right=3, "t"'] & A_0 \arrow[l, squiggly, "i" description]
\end{tikzcd}$$

(+ more maps)

Examples of $\mathbb{F}$-sketches

① C-sketch for pseudocats in an $\mathbb{F}$-cat:

$$\boxed{A_1 \times_{A_0} A_1 \overset{\xrightarrow{\pi_1}}{\underset{\xrightarrow{\pi_2}}{\rightsquigarrow_m}} A_1 \overset{\xrightarrow{s}}{\underset{\xrightarrow{t}}{\overset{i}{\leftsquigarrow}}} A_0}$$

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- Tight pullback sketch

Examples of $\mathbb{F}$-sketches

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(+ more maps)

- Tight pullback sketch
- Models in $\mathbf{Cat}$ (tight = loose, limit cones)

$\equiv$ pseudo-double cats

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(+ more maps)

- Tight pullback sketch
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$\equiv$ pseudo-double cats

- $\_\ \_\ \_\ \_\ \_$ $\mathbf{MonCat}_{s,p}$ (strict & strong)

$\equiv$ monoidal double cats

Examples of $\mathcal{F}$-sketches

② Similarly, $\mathbf{M}$ = sketch for pseudomonoids in an $\mathcal{F}$-category:

$$\boxed{\begin{tikzcd} A \times A \arrow[r, shift left=2, "\pi_1"] \arrow[r, squiggly, shift right=1, "m", "\pi_2"'] & A \arrow[r, shift left=2, "!"] & 1 \arrow[l, squiggly, shift left=1, "i"] \end{tikzcd}}$$

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Examples of $\mathcal{F}$-sketches

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- Tight product sketch
- Capture monoidal cats, monoidal double cats etc ....

Examples of $\mathcal{F}$-sketches

③ $\mathcal{F}$ = sketch for fibrations in an $\mathcal{F}$-category

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$$A \xrightarrow{f} B$$

Examples of $\mathcal{F}$-sketches

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Fibrations in an $\mathcal{F}$-category

$$
\begin{tikzcd}
A^2 \arrow[d, bend right=50, "\pi_2"'] \arrow[d, bend left=50, "\pi_1"] \arrow[d, phantom, "\Leftarrow\eta"] & & B/F \arrow[dll, "\pi_1"'] \arrow[d, "\pi_2"] \arrow[dl, phantom, "\lambda\Swarrow"] \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}
$$

Examples of $\mathcal{F}$-sketches

③ $\mathcal{F}$ = sketch for fibrations in an $\mathcal{F}$-category

$$\begin{tikzcd} A^2 \arrow[r, "k_f"] \arrow[d, bend right, "\pi_2"'] \arrow[d, bend left, "\pi_1"] \arrow[d, phantom, "\Leftarrow\alpha"] & B/f \arrow[d, "\pi_2"] \arrow[dl, "\pi_1"'] \arrow[d, phantom, shift right=3ex, "\lambda \Swarrow"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

$k_f(a \xrightarrow{\alpha} b) = (fa \xrightarrow{f\alpha} fb, b)$

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$$
\begin{tikzcd}
A^{2} \arrow[rr, "k_f"] \arrow[d, shift right=3, "\pi_2"'] \arrow[d, shift left=3, "\pi_1"] \arrow[d, phantom, "\Leftarrow\kappa"] & \arrow[d, phantom, "\dashv"] & B/f \arrow[ll, squiggly, bend right, "r_f"'] \arrow[dll, "\pi_1"'] \arrow[d, "\pi_2"] \arrow[dl, phantom, "\lambda\Downarrow"] \\
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$$

$\sim$ looze

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Examples of $\mathcal{F}$-sketches

③ $\mathcal{F}$ = sketch for

Fibrations in an $\mathcal{F}$-category

~ looze

$$
\begin{tikzcd}
A^2 \arrow[r, shift right, "k_F"'] \arrow[d, bend right, "\pi_2"'] \arrow[d, bend left, "\pi_1"] \arrow[d, phantom, "\overset{\eta}{\Leftarrow}"] & B/F \arrow[l, squiggly, shift right=2, "r_F"'] \arrow[l, phantom, "\top"] \arrow[dl, "\pi_1"'] \arrow[d, "\pi_2"] \\
A \arrow[r, "f"'] \arrow[ur, phantom, "\lambda\, \Swarrow", shift left=3] & B
\end{tikzcd}
$$

$k_F(a \xrightarrow{\alpha} b) = (fa \xrightarrow{F\alpha} fb, b)$

- tight comma object sketch

Examples of $\mathcal{F}$-sketches

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fibrations in an $\mathcal{F}$-category

$$\begin{tikzcd}
A^2 \arrow[r, "k_F"] \arrow[d, shift right=2, "\pi_2"'] \arrow[d, shift left=2, "\pi_1"] \arrow[d, phantom, "\Leftarrow\alpha"] & B/F \arrow[l, bend right, squiggly, "r_F\ \sim\ \text{loose}"'] \arrow[l, phantom, bend right=20, "\top"] \arrow[dl, "\pi_1"'] \arrow[d, "\pi_2"] \arrow[dl, phantom, shift left=3, "\lambda \Downarrow"] \\
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\end{tikzcd}$$

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(Shulman, Moeller-Vasilakopoulou)

Examples of $\mathcal{F}$-sketches

③ $\mathcal{F}$ = sketch for

Fibrations in an $\mathcal{F}$-category

$$
\boxed{
\begin{tikzcd}
A^2 \arrow[r, "k_f"] \arrow[d, bend right=40, "\pi_2"'] \arrow[d, bend left=40, "\pi_1"] \arrow[d, phantom, "\overset{\kappa}{\Leftarrow}"] & B/F \arrow[l, bend right, squiggly, "r_f"'] \arrow[l, phantom, "\top"] \arrow[dl, "\pi_1"'] \arrow[d, "\pi_2"] \arrow[dl, phantom, shift left=3, "\lambda \Swarrow"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
}
$$

~ loose

$k_f(a \xrightarrow{\kappa} b) = (fa \xrightarrow{f\kappa} fb, b)$

- tight comma object sketch
- models in $\mathcal{C}at$ - (cloven) fibrations
- in $\mathrm{MonCat}_{s,p}$ - monoidal fibrations:

  (Shulman, Moeller-Vasilakopoulou)
- in $\mathbb{D}\mathrm{bl}_{s,p}$ - double fibrations

  (Cruttwell, Lambert, Pronk, Szyld)

Models in models

Notation) For $\omega \in \{s, p, l, c\}$ let $\bar{s} = s$, $\bar{p} = p$, $\bar{l} = c$, $\bar{c} = l$.

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Theorem
Let $S, T, \mathcal{C}$ be $\mathcal{F}$-sketches.
Then we have iso of $\mathcal{F}$-sketches

$$\mathrm{Mod}_\omega(S, \mathrm{Mod}_{\bar{\omega}}(T, \mathcal{C})) \cong \mathrm{Mod}_{\bar{\omega}}(T, \mathrm{Mod}_\omega(S, \mathcal{C})).$$

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Proof

Biclosed monoidal cat of $\mathcal{F}$-sketches.

Applications

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eg. $\mathbf{C}, \mathbf{C}, e \to$ intercats (Grandis–Paré)

(pseudocats in Dbl_e $\equiv$ pseudocats in Dbl_c)

Applications

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eg. $\mathbf{C}, \mathbf{C}, e$ + intercats (Grandis-Paré)

(pseudocats in Dbl_e $\equiv$ pseudocats in Dbl_c)

- monoidal fibrations, monoidal double cats double fibrations, duoidal cats ...

Further applications

- Clarifies study of 2-d limit theories (Power & Lack / JB)

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- Clarifies study of 2-d limit theories (Power & Lack / JB)

Forthcoming paper:

Enhanced 2-dimensional limit sketches & double cats with structure (Arkor, Bourke, Ko)

References

- Lack-Shulman : Enhanced 2-cats ...
- Lack : Two-dimensional Lawvere theories (Talk @ CT 2009)
- Framed bicats & monoidal fibrations - Shulman
- Monoidal Grothendieck (Moeller Vasilakopoulou)
- Double Fibrations (CLPS)
- Accessible aspects of 2-cat th. (JB)