

Two-dimensional
limit theories
(enhanced)

John Bourke
(with N. Arkor , J. Ko)

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- Limit sketches (or theories)
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- Natural generalisation - 2-d limit sketches (or theories) - capture categorical structures like monoidal cats, double cats etc
- But problems emerge :
need to consider 2-cats with special maps (extra structure)
→ enhanced 2-d lim. sketches

Internal categories

- Internal category is a diagram

$$A_1 \times_{A_0} A_1 \begin{array}{c} \xrightarrow{\pi_1} \\ \xrightarrow{m} \\ \xrightarrow{\pi_2} \end{array} A_1 \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{i} \\ \xrightarrow{t} \end{array} A_0$$

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- Internal functor $F: A \rightarrow B$:

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 F_2 = F_1 \times_{F_0} F_1 \downarrow & & \downarrow F_1 & & \downarrow F_0 \\
 B_1 \times_{B_0} B_1 & \begin{array}{c} \xrightarrow{\pi_1} \\ \xrightarrow{m} \\ \xrightarrow{\pi_2} \end{array} & B_1 & \begin{array}{c} \xleftarrow{s} \\ \xleftarrow{i} \\ \xrightarrow{t} \end{array} & B_0
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& strong monoidal functors

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- Or a pseudomonoid

$$A^2 \begin{array}{c} \xrightarrow{\pi_1} \\ \xrightarrow{m} \\ \xrightarrow{\pi_2} \end{array} A \xleftarrow{t} 1$$

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Summary so Far

Need to move from diag like

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- → enhanced Z-cat theory & enhanced Z-d limit sketches

Enhanced 2-categories

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- Induces a pair of 2-cats $\mathcal{A}_\tau, \mathcal{A}_\lambda$ connected by 2-functor $J: \mathcal{A}_\tau \hookrightarrow \mathcal{A}_\lambda$
 $\begin{array}{ccc} & \downarrow \text{tight maps} & \downarrow \text{loose maps} \\ & \mathcal{A}_\tau & \mathcal{A}_\lambda \end{array}$

- Many examples: -

$\text{MonCat}_{s,l}$: monoidal cats, strict & lax mon. functors

- or any combination.

As \mathcal{F} -enriched cats

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• \mathcal{F} -functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is a

\mathcal{Z} -functor $F: \mathcal{A}_\lambda \rightarrow \mathcal{B}_\lambda$ preserving tightness

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• \mathcal{F} -natural transformation =
2-nat transf. with tight components.

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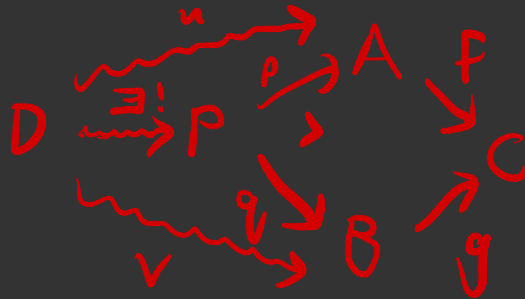
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- E.g. Tight pullbacks



Enriched limit sketches

- A V-limit sketch is a V-cat ^{Kelly 1982}

T with a collection of

$(W: J \rightarrow V, D: J \rightarrow \mathcal{C}, p: W \rightarrow \mathcal{C}(X, D-))$
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- $V = \text{Cat} \Rightarrow$ Two-dim. lim sketch

Enriched limit sketches

- A U -limit sketch is a U -cat

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- $U = \text{Cat} \rightarrow$ Two-dim. lim sketch
- $U = \mathbb{F} \rightarrow$ enhanced 2-d lim sketch

Enriched limit sketches

Kelly 1982

- A V-limit sketch is a V-cat T with a collection of $(W: J \rightarrow V, D: J \rightarrow \mathcal{C}, p: W \rightarrow \mathcal{C}(X, D-))$
weights *diagrams* *weighted cones*
- $V = \text{Cat} \rightarrow$ Two-dim. lim sketch
- $V = \mathcal{F} \rightarrow$ enhanced 2-d lim sketch
- Model of T in V-sketch \mathcal{C} is a sketch morphism:
V-functor preserving weighted cones.

Models of \mathcal{F} -sketches

$\text{Mod}_{\text{lax}}(T, \mathcal{C})$ is \mathcal{F} -cut of models
& lax maps

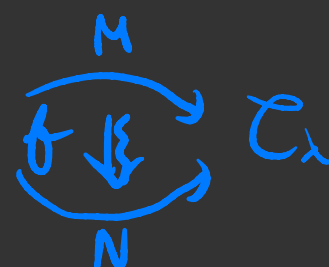
Models of \mathcal{F} -sketches

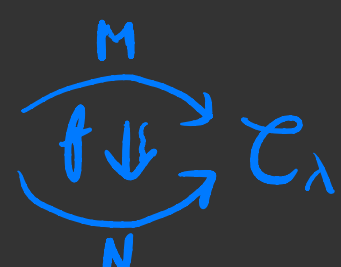
$\text{Mod}_{\text{Lax}}(\mathcal{T}, \mathcal{C})$ is \mathcal{F} -cat of models
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- loose morph $M \rightsquigarrow N$ is $\begin{array}{ccc} & M & \\ & \downarrow f & \\ & N & \end{array} \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array} \mathcal{C}_\lambda$
Lax natural transf T_λ such that
such that $T_\mu \xrightarrow{J} T_\lambda$ $\begin{array}{ccc} & M & \\ & \downarrow f & \\ & N & \end{array} \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array} \mathcal{C}_\lambda$ is \mathcal{Z} -natural :

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ie.
$$\begin{array}{ccc} MX & \xrightarrow{f_x} & NX \\ Mx \downarrow & f_x \Downarrow & \downarrow Nx \\ MY & \xrightarrow{f_y} & NY \end{array}$$
 is an identity 2-cell
for $\alpha: X \rightarrow Y$ tight.

Models of \mathcal{F} -sketches

$\text{Mod}_{\lambda}(T, \mathcal{C})$ is \mathcal{F} -cat of models
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• tight = \mathcal{F} -nat transformation

Models of \mathcal{F} -sketches

$\text{Mod}_{\lambda}(T, \mathcal{C})$ is \mathcal{F} -cat of models
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• loose morph $M \rightsquigarrow N$ is \mathcal{M}
lax natural transf $T_{\lambda} \begin{array}{c} \mathcal{M} \\ \Downarrow \\ \mathcal{C}_{\lambda} \end{array}$

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• tight = \mathcal{F} -nat transformation

• \mathcal{Z} -cells are modifications

F-cats of models ctd

• $\omega \in \{s, p, l, c\}$

strict

pseudo

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• F-cat of models $\text{Mod}_{\omega}(T, \mathcal{C})$

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• Each in fact an F-sketch :
take levelwise weighted cones.

Examples of \mathcal{F} -sketches

① C - sketch for pseudocats in an \mathcal{F} -cat :

$$A_1 \times_{A_0} A_1 \begin{array}{c} \xrightarrow{\pi_1} \\ \xrightarrow{m} \\ \xrightarrow{\pi_2} \end{array} A_1 \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{t} \\ \xrightarrow{t} \end{array} A_0 \quad \left(\begin{array}{l} + \text{ more} \\ \text{maps} \end{array} \right)$$

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- Tight pullback sketch

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 - Models in Cat $\left(\begin{array}{l} \text{tight} = \text{loose,} \\ \text{limit cones} \end{array} \right)$
- \equiv pseudo-double cats

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- ≡ pseudo-double cats
- MonCat_{s,p} $\left(\begin{array}{l} \text{strict \&} \\ \text{strong} \end{array} \right)$
- ≡ monoidal double cats

Examples of \mathcal{F} -sketches

② Similarly, \mathcal{M} = sketch for pseudomonoids in an \mathcal{F} -category:

$$A \times A \begin{array}{c} \xrightarrow{\pi_1} \\ \xrightarrow{\quad m \quad} \\ \xrightarrow{\quad \pi_2 \quad} \end{array} A \begin{array}{c} \xleftarrow{!} \\ \xleftarrow{\quad \mu \quad} \\ \xleftarrow{\quad ! \quad} \end{array} 1$$

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- Tight product sketch
- Capture monoidal cats,
monoidal double cats
etc ...

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$$A \xrightarrow[\mathcal{F}]{} B$$

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$$\begin{array}{ccc} A^{\mathcal{Z}} & & B/F \\ \pi_2 \circ \pi_1 \swarrow \pi & & \downarrow \pi_2 \\ A & \xrightarrow{f} & B \end{array}$$

Examples of \mathcal{F} -sketches

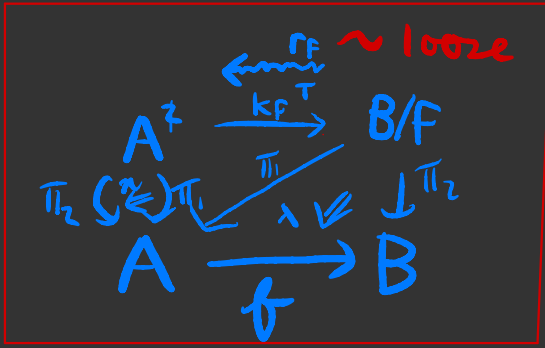
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$$k_{\mathcal{F}}(a \xrightarrow{x} b) = (fa \xrightarrow{fx} fb, b)$$

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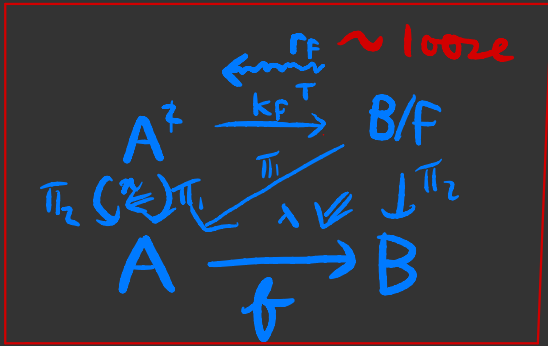
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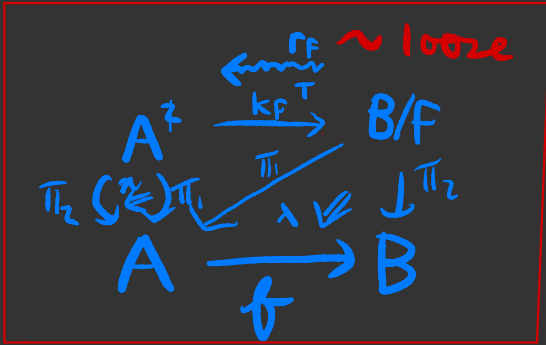
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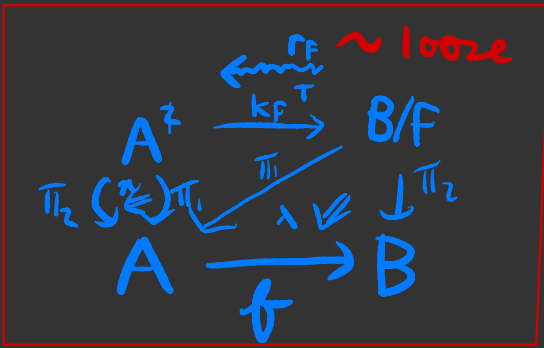
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- models in Cat - (cloven) fibrations

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- tight comma object sketch
- models in Cat - (cloven) fibrations
- in $\text{MonCat}_{s,p}$ - monoidal fibrations:
(Shulman, Moeller - Varilakopoulou)
- in $\text{Dbl}_{s,p}$ - double fibrations
(Crutwell, Lambert, Pronk, Szlyd)

Models in models

Notation) For $\omega \in \{s, p, l, c\}$

let $\bar{s} = s, \bar{p} = p, \bar{l} = c, \bar{c} = l$.

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Theorem

Let S, T, \mathcal{C} be \mathcal{F} -sketches.

Then we have iso of \mathcal{F} -sketches

$$\text{Mod}_\omega(S, \text{Mod}_{\bar{\omega}}(T, \mathcal{C})) \cong \text{Mod}_{\bar{\omega}}(T, \text{Mod}_\omega(S, \mathcal{C})).$$

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Proof

Biclosed monoidal cat of
 \mathcal{F} -sketches.

Applications

$$\text{Mod}_w(S, \text{Mod}_{\bar{w}}(T, \mathcal{C})) \cong \text{Mod}_{\bar{w}}(T, \text{Mod}_w(S, \mathcal{C}))$$

- Mixing sketches C, M, F & weaknesses w explains different perspectives on structures in literature

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eg. $C, C, e \rightarrow \text{interacts}$ (Grandis-Pare)
(pseudocats in $\text{Dbl}_e \equiv \text{pseudocats in } \text{Dbl}_c$)

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- monoidal fibrations,
monoidal double cats
double fibrations,
duoidal cats - - -

Further applications

- Clarifies study of
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(Power & Lack / JB)

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Forthcoming paper :

Enhanced Z-dimensional limit
sketches & double cuts
with structure

(Arkar, Bourke, Ko)

References

- Lack-Shulman: Enhanced 2-cats ...
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